Central Force Motion: The Kepler Problem

The orbital motion of a planet around the Sun was one of the first important problems to be analyzed in terms of Newton’s three laws. The gravitational force attracting a planet to the Sun is a central force, that is, a force directed towards a fixed point. The motion of a planet is a prime example of the more general problem of the behavior of a particle acted upon by a central force.

Although we shall be primarily concerned with the motion of planets and satellites, the techniques you will learn in this chapter are applicable to any kind of central force. In this chapter, as well as learning the laws governing the motion of celestial bodies, you will be exposed to the concept of effective potential energy, and how constants of the motion are used in solving physics problems.

Historically, the quantitative analysis of orbital motion began with Kepler’s realization that the motion of planets can be described by three empirical laws. An important point made in this chapter is that Kepler’s laws of planetary motion can be derived theoretically from Newton’s laws of motion. Additionally, Newton’s laws give us a much deeper understanding of Kepler’s laws. This application of Newton’s ideas amazed and fascinated the “natural philosophers” of his era and was one of the most important events in the history of science.\(^1\)

10. CENTRAL FORCE MOTION: THE KEPLER PROBLEM

10.1. Johannes Kepler (Optional Historical Note)

Johannes Kepler lived from 1571 to 1630. He was born into a poor family in a small town in Germany. His father was a professional soldier who spent much of his time away from home and his mother was a quarrelsome woman who was accused of witchcraft in her old age. It would be fair to assume that Kepler had an unhappy childhood. Nevertheless, he was very intelligent and an excellent student. The prince who ruled the region sent him to study at a Lutheran seminary. Eventually, Kepler graduated from the University of Tubingen. Originally he planned to study divinity, but before he was ordained, the authorities in the seminary convinced him that he was not cut out for the clergy. With the help of his advisers, he was appointed to a position teaching mathematics in Graz, Austria. He was a quiet, introspective person and not greatly interested in teaching; in fact, he was probably a terrible teacher for he had the reputation of interrupting his own lectures to silently mull over some idea that had just occurred to him.

In Graz, Kepler had an idea that changed the course of his life. This idea, which became an obsession with him, was (he thought) a glimpse into the mind of God: A vision of the basic structure of the universe. It was, so to speak, revealed to him why there are only five planets, and why they are in their particular orbits around the sun. Kepler’s mind-boggling inspiration was this: there are only five planets (besides Earth) because there are only five “perfect” solids, and the orbits of the planets correspond to spheres circumscribed about the perfect solids when they are nested, one inside the other.

The “perfect” solids (or “simple polyhedrons”) are the geometrical figures formed from regular polygons. A regular polygon is one with equal sides. Thus, for example, a tetrahedron (or equilateral pyramid) is made up of four equilateral triangles. A cube is made up of six squares. Similarly, an octahedron is an eight sided solid made of equilateral triangles, an icosahedron is made of twenty equilateral triangles and the dodecahedron is composed of twelve pentagons. Many other solids can be constructed from polygons; for example a “soccer ball” shape can be made up of pentagons and hexagons. This is the structure of Carbon-60, the so-called “Bucky balls,” named in honor of Buckminster Fuller who studied the properties of such structures. There are, however, only five “perfect” solids whose faces are a single type of regular polygon. There is a very neat proof that there can only be five such solids; this proof can be traced back to the ancient Greeks.
If you are interested, it is reproduced in the book *Cosmos*, by Carl Sagan.\(^2\)

Kepler’s idea was that if the perfect solids were placed one inside the other, and each was circumscribed with a sphere, then the sun would be at the center of the system and each of the five planets would orbit the sun in a circular orbit whose radius would be equal to the radius of the corresponding circumscribed sphere. Kepler built models of these nested solids and their circumscribed spheres, but he could not prove his theory because he did not have enough information on the distances of the five (known) planets to the Sun. Figure 10.1 illustrates the idea behind Kepler’s model.

![Figure 10.1. A nested tetrahedron and cube with inscribed and circumscribed spheres. In Kepler’s model the nested perfect solids (from innermost outwards) were an octahedron, an icosahedron, a dodecahedron, a tetrahedron and a cube.](image)

At that time, the best astronomical data in Europe were in the observatory of an eccentric Danish astronomer named Tycho Brahe.\(^3\)


\(^{3}\)Tycho Brahe (1546-1601) set up an observatory at Uraniborg on the Danish island of Hven. A stream of young assistants from all over Europe came to Uraniborg where they carried out experiments in chemistry during the day and observed the heavens at night. Uraniborg has been described as the first research institution involved in “big science.” For the full story read the book by John Robert Christianson, *On Tycho’s Island: Tycho Brahe and his Assistants, 1570-1601*, Cambridge University Press, Cambridge, 2000. At the time Kepler went to visit him, Brahe had moved to Prague.
Kepler went to visit Brahe to get his data. He was certain that Brahe would be overwhelmed by his wonderful new theory. Brahe was not overwhelmed. In fact, at first Brahe would not even let Kepler see the data! However, Brahe soon realized that Kepler was an excellent mathematician and he offered to let him work on a small portion of the data to calculate the orbit of Mars. This was certainly not what Kepler had in mind, but he grudgingly agreed.

This was, perhaps, the original “graduate student - professor” relationship in science. To this day the same pattern exists. A young, aspiring scientist makes the pilgrimage to the laboratory of the established professor with the hope of being allowed to share in the professor’s knowledge and data. You yourself may be doing this a few years from now. I hope your relationship with your graduate advisor will less tempestuous than that of Kepler and Tycho. They did not get along at all. Tycho was a man who loved a party, who spent his evenings eating, drinking, and carousing, whereas Kepler was somber and rather puritanical and did not at all approve of Tycho’s lifestyle.

Some years after the collaboration began, Tycho died. (Rumor has it that he died of a burst bladder while on a drinking spree.) Kepler inherited all of Tycho’s data as well as Tycho’s position as court astronomer. After a great deal of analysis, much to his dismay, Kepler found that Tycho’s data did not support his grandiose theory. In fact, the data showed that the orbits of planets were not even circles, but rather ellipses! Kepler tried to modify his theory by slipping elliptical orbits between the inscribed perfect solids, but it did not quite work. Kepler never did know that there are more than five planets, as the discoveries of the planets Uranus and Neptune came many years after his death. By the time these were discovered, Kepler’s theory on the inscribed orbits of the five planets was no more than a historical oddity.

Kepler’s life is full of instructive incidents for physicists. His most valuable contribution to science was his analysis and synthesis of the observations of Tycho Brahe. It is interesting to consider that he was obsessed by a beautiful theory that did not agree with experiment. No matter how beautiful a theory may be, if it does not agree with experimental measurements, it must be discarded! As a physicist you must never let your theories carry you away. Physics is the study of the physical universe and it is Nature that determines the way things behave. It was to Kepler’s great credit that he respected and believed the data, even though the data did not agree with his theory.
10.2. Kepler’s Laws

Kepler’s synthesis of Tycho’s planetary data can be expressed as three statements that are now called Kepler’s laws of planetary motion. They are:

1. The orbit of a planet is an ellipse with the sun at one focus.
2. The radius vector of a planet (the sun-planet line) sweeps out equal areas in equal times.
3. The square of the period of a planet is proportional to the cube of the semi-major axis of its elliptical orbit.

These laws express three facts about the motion of a planet. They are “empirical laws,” that is, they were obtained from the data but they had no theoretical basis. Kepler’s laws describe the motion of planets but neither Kepler nor anyone else living at that time could give a reasonable explanation why planets behave in this manner. About 70 years later, Newton applied the law of universal gravitation and his laws of motion to the problem and succeeded brilliantly in showing that Kepler’s laws are a consequence of some very basic physical relations. It is no wonder that other scientists of his era were in such awe of him. In this chapter, I will show you what Newton did, but of course I will use modern methods. Near the end of the chapter I will return to Kepler’s laws: by that time you should have a deeper appreciation for them.

10.3. Central Forces

Any force that is directed toward or away from a fixed point (often taken as the origin of coordinates) and whose magnitude is a function only of the distance to the fixed point is called a central force. In general, a central force will have the form

$$\mathbf{F} = f(r)\hat{r},$$

where \( f(r) \) is the magnitude of the force (a function only of \( r \)) and \( \hat{r} \) is the radial unit vector. The direction of the force is along the line joining the particle and the origin. Two important central forces are the gravitational force between masses and the electrostatic force between charges. Assuming one of the bodies is fixed at the origin, the gravitational force is a central force given by

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{r},$$

and the electrostatic force is

$$\mathbf{F} = \frac{Q_1Q_2}{4\pi\varepsilon_0r^2}\hat{r}.$$
We can also imagine other central forces such as, for example,
\[ \mathbf{F} = \frac{k}{r^5} \hat{r}. \]
Such forces may or may not exist in nature, but we can analyze them mathematically anyway. Although this seems like a useless theoretical exercise, such studies may have a practical outcome. For example, to a first approximation, the force between molecules can be expressed as the sum of two central forces. This “Lennard-Jones” force can be expressed in terms of the potential energy as
\[ V(r) = -\frac{a}{r^6} + \frac{b}{r^{12}}. \]
Here \( a \) and \( b \) are constants that depend on the properties of the particular molecules involved in the interaction. The force itself is obtained from \( \mathbf{F} = -\nabla V \).

We are interested in determining the motion of a particle subjected to the gravitational force of a second body (which we also treat as if it were a particle). For the sake of simplicity let us assume the second body is at rest. Strictly speaking, this cannot be true; the two objects actually orbit about their center of mass. But if one particle is much more massive than the other, then the more massive particle has a much smaller acceleration and in the limit, as the ratio of the masses goes to infinity, the more massive particle can be considered to be at rest.

To appreciate this, imagine a binary star system in which both stars have the same mass. The two stars will orbit around their center of mass, which is at a point halfway between them. But suppose one of the stars is more massive than the other. The center of mass will be closer to the more massive star. If one star is infinitely more massive than the other, then the center of mass will be at the center of the larger star. Similarly, the space shuttle and the Earth are orbiting around their center of mass, but for all intents and purposes, the center of mass of this system lies at the center of the Earth and we are perfectly justified in considering the Earth to be at rest.\(^4\)

In considering the motion of a particle of mass \( m \) in a central force field, the first thing to note is that the particle moves in a plane. The proof of this statement is given by the following argument. At any

\(^4\)Recall the discussion of reduced mass, defined as \( \mu = \frac{m_1 m_2}{m_1 + m_2} \). See Equation (6.20). If the central force problem is treated in terms of the reduced mass, then no approximation is made. The relations obtained are the same as those derived here except that \( m \) is replaced by \( \mu \) and \( \mathbf{r} \) is the position of one body relative to the other rather than the distance from the center of mass.
instant in time the particle is in the plane defined by the position vector and the velocity vector. The only way the particle can move out of this plane is if its acceleration has a component perpendicular to the plane. But for a central force, the force (and hence the acceleration) lies along $\mathbf{r}$, and therefore there is no component of the acceleration perpendicular to the plane.

Perhaps the most important aspect of the motion of a particle under the action of a central force is that the moving particle has a constant angular momentum. By definition, the angular momentum of a particle is $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ where $\mathbf{r}$ is the position of the particle and $\mathbf{p}$ is its linear momentum. Recall from Section 7.2.2 that the time derivative of the angular momentum is

$$\frac{d\mathbf{l}}{dt} = \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F}.$$  

In the case of a central force, $\mathbf{F} = f(r)\mathbf{\hat{r}}$. Hence,

$$\frac{d\mathbf{l}}{dt} = \mathbf{r} \times f(r)\mathbf{\hat{r}} = r\mathbf{\hat{r}} \times f(r)\mathbf{\hat{r}} = rf(r)(\mathbf{\hat{r}} \times \mathbf{\hat{r}}) = 0.$$  

Consequently, for a central force, the time derivative of the angular momentum is zero. This means that the angular momentum is constant. Another way of looking at this is to recall that the time rate of change of angular momentum is equal to the torque. A central force cannot exert a torque on a particle. You should convince yourself that this statement is true.

These simple physical arguments lead us to an important conservation law concerning the motion of any particle under the action of any central force:

**The angular momentum is constant.**

This means that both the magnitude and the direction of the angular momentum are constant.

For a more sophisticated proof of the constancy of angular momentum, you can write the Lagrangian for a particle in a central force field. In polar coordinates the kinetic energy is $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ and the potential energy is $V = -\int_{r_0}^{r} f(r)dr$. The Lagrangian is

$$L = T - V = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \int_{r_0}^{r} f(r)dr.$$  

Since $\theta$ is ignorable, $\partial L/\partial \dot{\theta} = \text{constant}$. That is,

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} = l = \text{constant}.$$ 

We can use the constancy of angular momentum to generate another proof that a particle in a central force field moves in a plane. Assume the particle is at position $\mathbf{r}$ and has velocity $\mathbf{v}$. These two vectors define a plane and, of course, the particle lies in this plane. By the definition of cross product, the vector $\mathbf{r} \times \mathbf{v}$ is perpendicular to the plane containing the vectors $\mathbf{r}$ and $\mathbf{v}$. But $\mathbf{l} = m\mathbf{r} \times \mathbf{v}$, so the vector $\mathbf{l}$ is perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{v}$. Since $\mathbf{l} = \text{constant}$, the perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{v}$ is constant. Therefore, the particle moves in a constant plane. See Figure 10.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10_2.png}
\caption{The angular momentum vector is perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{v}$. Since the angular momentum is constant, the plane is invariant.}
\end{figure}

Since the motion of the particle lies in a plane, two coordinates are sufficient to specify its position. These can be $x$ and $y$ or $r$ and $\theta$. The origin of the coordinate system is usually placed at the primary (assumed to be at rest).\(^5\) To orient the coordinates, it is necessary to specify a fixed direction. Astronomers pick an imaginary line from the center of the Earth towards a position called "The First Point in Aries" which is the position of the Sun at the vernal equinox.

In Figure 10.3 the symbol $\Upsilon$ indicates the fixed line in space. In polar coordinates the angle $\theta$ is measured from $\Upsilon$. In Cartesian coordinates one usually defines the $x$-axis along this fixed line. The $y$-axis is selected in the plane of the orbit and perpendicular to the $x$-axis. The $z$-axis is then perpendicular to the orbit and along the angular momentum vector. For many problems it is quite safe to assume that the coordinate system is an inertial system.

\(^5\)In the language of astronomy, if one body is much more massive than the other, the massive body is called the "primary." The primary is often assumed to be at rest. Actually, of course, both bodies move about the center of mass.
Five thousand years ago the position of the Sun at noon on the vernal equinox was in the constellation Aries ("the ram"). Since the Earth’s axis of rotation precesses with a period of about 25,000 years, the position of the Sun at noon on the vernal equinox has changed and it is presently in the constellation Pisces; within a few hundred years it will enter into the constellation Aquarius. However, the name “First Point in Aries” and the symbol $\Theta$, representing ram’s horns, are still used to represent this arbitrary fixed line in space. It is amusing to note that due to the precession of the Earth’s axis, the positions of the zodiacal constellations have shifted, but astrologers still use the values of 5000 years ago. Thus, people born when the Sun was in the constellation Pisces think they are Aries, those who were born when the Sun was in Aries think they are Taurus, and so on. If astrology had any validity, this horrible mix-up in the zodiacal signs would be serious indeed!

**Worked Example 10.1.** A particle is in a circular orbit under the action of an attractive central force given by $f(r) = -k/r^3$. Obtain an expression for the angular momentum and show that it is constant.

**Solution:** By Newton’s second law, $m\ddot{\mathbf{r}} = -f(r)\hat{\mathbf{r}} = -\frac{k}{r^3}\hat{\mathbf{r}}$. Recall from our discussion of plane polar coordinates in Chapter 2 that the acceleration in polar coordinates is

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{\theta}}.$$  

For a circular orbit, the magnitude $r$ is constant, so $\dot{r} = 0$ and $\ddot{r} = 0$ so

$$m\ddot{\mathbf{r}} = m(-r\dot{\theta}^2\hat{\mathbf{r}} + r\ddot{\theta}\hat{\mathbf{\theta}}) = -f(r)\hat{\mathbf{r}}.$$
Therefore, for an inverse cube force law, \( m\ddot{r} = -f(r)\hat{r} \) yields two relations, namely
\[
-mr\dot{\theta}^2 = -f(r) = -\frac{k}{r^3},
\]
and
\[
mr\ddot{\theta} = 0.
\]
The second equation boils down to \( \ddot{\theta} = 0 \), telling us that the angular velocity, \( \dot{\theta} \) is constant. The first equation yields
\[
mr^4\dot{\theta}^2 = k.
\]
But since \( l = mr^2\dot{\theta} \) we see that the left hand side is \( l^2/m \) and consequently
\[
l^2 = mk,
\]
and hence the angular momentum is constant. Actually, this result also follows from \( r = \text{constant} \) and \( \dot{\theta} = \text{constant} \).

**Worked Example 10.2.** Consider a particle in an attractive force field whose potential energy is of the form \( V(r) = kr^{n+1} \). Show that for a periodic orbit the average kinetic energy is related to the average potential energy by
\[
\langle T \rangle = \frac{n + 1}{2} \langle V \rangle.
\]
Apply to the gravitational force. (This is a special case of the virial theorem.)

**Solution:** This problem asks us to determine the relationship between the average values of the kinetic and potential energy. The time average of any quantity (such as the kinetic energy) is defined thus:
\[
\langle T \rangle = \frac{1}{\tau} \int_0^\tau T(t) dt.
\]
For this problem, it is reasonable to let \( \tau \) be the orbit period.

If a particle is in orbit, after one period it will have returned to its original location and have the same velocity as it did initially. Therefore, the quantity \( \mathbf{p} \cdot \mathbf{r} \) repeats periodically. Let us define the periodic function \( G(t) = \mathbf{p} \cdot \mathbf{r} \). Consider the time average of the derivative of \( G \) with respect to time:
\[
\left\langle \frac{dG}{dt} \right\rangle = \frac{1}{\tau} \int_0^\tau \frac{dG}{dt} dt = \frac{1}{\tau} \int_0^\tau dG = \frac{1}{\tau} [G(\tau) - G(0)] = 0.
\]
But
\[
\frac{dG}{dt} = \frac{d}{dt} (\mathbf{p} \cdot \mathbf{r}) = \mathbf{p} \cdot \dot{\mathbf{r}} + \dot{\mathbf{p}} \cdot \mathbf{r} = m \mathbf{v} \cdot \mathbf{v} + \mathbf{F} \cdot \mathbf{r} = 2T - \nabla V \cdot \mathbf{r}
\]
\[
= 2T - \frac{dV}{dr} r,
\]
because \(\nabla V \cdot \mathbf{r} = \frac{dV(r)}{dr} r\). Therefore,
\[
\langle \frac{dG}{dt} \rangle = \frac{1}{\tau} \int_0^\tau \left( 2T - \frac{dV}{dr} r \right) dt = 0,
\]
and
\[
2 \langle T \rangle - \langle \frac{dV}{dr} r \rangle = 0,
\]
\[
\langle T \rangle = \frac{1}{2} \langle \frac{dV}{dr} r \rangle.
\]
Since \(V = kr^{n+1}\) we have
\[
\frac{dV}{dr} r = (n + 1)kr^{n}r = (n + 1)kr^{n+1} = (n + 1)V,
\]
and
\[
\langle \frac{dV}{dr} r \rangle = (n + 1) \langle V \rangle,
\]
so
\[
\langle T \rangle = \frac{n + 1}{2} \langle V \rangle.
\]
The gravitational potential has the form \(V = -\frac{k}{r}\) so \(n = -2\) and
\[
\langle T \rangle = -\frac{1}{2} \langle V \rangle.
\]
This relationship between kinetic and potential energy is useful when solving orbital mechanics problems. It might be mentioned that the virial theorem has important applications in thermodynamics and statistical mechanics.

**Exercise 10.1.** Use \(\mathbf{F} = -\nabla V\) to obtain an expression for the Lennard-Jones force. Determine the value of \(r\) where the force changes from attractive to repulsive. Answer: \(\mathbf{F} = -\mathbf{F} \left( \frac{6a}{r^6} - \frac{12b}{r^2} \right)\).

**Exercise 10.2.** Use \(V = -\int_{r_0}^r F(r)dr\) to obtain the potential energy for the gravitational force, the electrostatic force, and the force exerted by a spring. Select \(r_0\) appropriately.
Exercise 10.3. Assume that at some initial moment the radius vector \( \mathbf{r} \) and the velocity vector \( \mathbf{v} \) are perpendicular to one another, however the angle between them is changing with time. Explain why the two vectors cannot ever be parallel to one another. Answer: Because angular momentum is constant.

10.4. The Equation of Motion

Our next task is to determine the equation of motion for a planet. We will do this in two ways: first by writing down Newton’s second law, and second by applying the Lagrangian technique.

The force acting on a particle (planet) of mass \( m \) is given by Newton’s law of universal gravitation

\[
\mathbf{F} = -\frac{GMm}{r^2} \hat{\mathbf{r}},
\]

where \( M \) is the mass of the larger attracting body, assumed fixed at the origin of coordinates. The equation of motion of the satellite (mass \( m \)) is

\[
m\ddot{\mathbf{r}} = -\frac{GMm}{r^2} \hat{\mathbf{r}},
\]

or

\[
\ddot{\mathbf{r}} = -\frac{GM}{r^2} \hat{\mathbf{r}}. \tag{10.1}
\]

We evaluated the acceleration \( \ddot{\mathbf{r}} \) in polar coordinates in Chapter 2 (see Equation 2.14), obtaining

\[
\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2) \hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{\mathbf{\theta}}.
\]

Substituting this expression into Equation (10.1), yields

\[
(\ddot{r} - r\dot{\theta}^2) \hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{\mathbf{\theta}} = -\frac{GM}{r^2} \hat{\mathbf{r}}.
\]

Separately equating the radial and angular components leads to the following two scalar equations

\[
\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}, \tag{10.2}
\]

\[
r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.
\]

These equations are a pair of coupled second order ordinary differential equations. The second equation, the “\( \dot{\theta} \)-equation,” is easy to analyze by going back to the definition of angular momentum and recalling that

\[
\mathbf{l} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}.
\]
The velocity in plane polar coordinates is $v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ (see Equation 2.13), so

$$l = m \left[ r\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \right].$$

Now carry out the cross products, using the facts that $\hat{r} \times \hat{r} = 0$ and $\hat{r} \times \hat{\theta} = \hat{k}$, where $\hat{k}$ is perpendicular to the plane of motion. You obtain

$$l = mr(r\dot{\theta})\hat{k} = mr^2\dot{\theta}\hat{k}. \quad (10.3)$$

This, in itself, is a useful equation. But it becomes even more useful if you take its time derivative,

$$\frac{dl}{dt} = \frac{d}{dt} \left( mr^2\dot{\theta} \right) \hat{k}.$$ 

For central forces, the angular momentum is constant, so $dl/dt = 0$ and the left hand side of this equation is zero. Therefore,

$$0 = \frac{d}{dt} \left( mr^2\dot{\theta} \right) = m \left( 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} \right),$$

$$= mr \left( 2\dot{r}\dot{\theta} + r\ddot{\theta} \right).$$

Since neither $m$ nor $r$ is equal to zero, this implies that

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$ 

But this is just the $\theta$ equation! Therefore, we see that the $\theta$ component of the equation of motion (the second of equations 10.2) is a statement that angular momentum is constant. Consequently, that equation can be replaced by the equivalent equation

$$l = mr^2\dot{\theta} = \text{constant}.$$ 

This equation, in turn, gives a nice expression for $\dot{\theta}$, namely,

$$\dot{\theta} = \frac{l}{mr^2}. \quad (10.4)$$

Replacing $\dot{\theta}$ in the $r$ equation (the first of Equations 10.2) by expression (10.4) yields the following equation for the radial motion of mass $m$:

$$\ddot{r} - \frac{l^2}{m^2r^3} = -\frac{GM}{r^2}. \quad (10.5)$$

This equation involves only $r$. Thus, conservation of angular momentum de-couples the equations of motion. The equation has only one variable ($r$), so it is often called a “one-dimensional equation.” But you should always keep in mind that the motion takes place in two dimensions.
Once Equation (10.5) has been solved for \( r = r(t) \), you can use it in Equation (10.4) to determine \( \theta = \theta(t) \).

We shall now use the Lagrangian technique to determine the equations of motion. (We had better obtain the same result!)

As you know, the Lagrangian is \( L = T - V \). In this problem \( V \) is the potential energy for a particle of mass \( m \) attracted gravitationally to a body of mass \( M \). According to Equation (9.6) this is

\[
V(r) = -\frac{GMm}{r}.
\]

The kinetic energy for this two-dimensional problem is \( T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \), or in polar coordinates,

\[
T = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\theta}^2.
\]

Therefore the Lagrangian is

\[
L = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{GMm}{r}.
\]

Recall that the Lagrange equations of motion have the form

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,
\]

where the \( q_i \)'s are now \( r \) and \( \theta \). So we have two equations, namely,

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0,
\]

and

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0.
\]

The partial derivatives are easily evaluated. You should prove for yourself that the two equations of motion are

\[
\frac{d}{dt} (m\dot{r}) - mr\dot{\theta}^2 + \frac{GMm}{r^2} = 0, \quad (10.6)
\]

and

\[
\frac{d}{dt} \left( mr^2\dot{\theta} \right) = 0, \quad \text{or} \quad \frac{dl}{dt} = 0. \quad (10.7)
\]

The second of these equations gives

\[
\dot{\theta} = \frac{l}{mr^2}.
\]

Using this expression, Equation (10.6) leads to

\[
\ddot{r} - \frac{l^2}{m^2r^3} = -\frac{GM}{r^2}. \quad (10.8)
\]
These are, of course, the same as the equations of motion obtained using Newton’s second law, namely Equations (10.4) and (10.5). Note, however, how much easier it is to use the Lagrangian than Newton’s second law.

**Exercise 10.4.** Carry out the steps to show that \( T = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\theta}^2 \).

**Exercise 10.5.** Obtain Equations (10.6) and (10.7).

**Exercise 10.6.** Suppose the force between a particle of mass \( m \) and a fixed point is given by \( F = -kr\mathbf{\hat{r}} \) where \( k \) is a constant. Obtain the Lagrangian and the equations of motion. Is angular momentum conserved for this system? Answer: \( m\ddot{r} - mr\dot{\theta}^2 + kr = 0; \frac{d}{dt}(mr^2\dot{\theta}) = 0 \).

### 10.5. Energy and the Effective Potential Energy

We have (twice!) obtained the equation of motion for the radial coordinate (Equation 10.5 and Equation 10.8). You probably expect to proceed by integrating it to get the value of \( r \) as a function of time and initial conditions. Indeed, we shall do that in Section 10.6, but first let us consider what the conservation of energy principle tells us about our problem. Since gravity is a conservative force, the total mechanical energy is constant. Thus,

\[
E = \text{constant} = T + V = \frac{1}{2}m(r^2 + r^2\dot{\theta}^2) - G\frac{Mm}{r}.
\]

But \( l = mr^2\dot{\theta} \) so we can replace \( \dot{\theta} \) by \( l/mr^2 \) and write

\[
E = \frac{1}{2}mr^2 + \frac{m}{2}r^2\left(\frac{l^2}{m^2r^4}\right) - G\frac{Mm}{r},
\]

or

\[
E = \frac{1}{2}mr^2 + \frac{l^2}{2mr^2} - G\frac{Mm}{r}.
\]

This looks like the relation \( E = T + V \) if we associate a “radial kinetic energy” with the term \( \frac{1}{2}mr^2 \) and an “effective potential energy” \( V_{\text{eff}} \).
with the remaining two terms. The effective potential $V_{eff}$ is given by

$$V_{eff} = \frac{l^2}{2mr^2} - \frac{GMm}{r}. \quad (10.10)$$

Although $V_{eff}$ looks and acts like a potential energy and is a function only of position, it is definitely not a potential energy since it actually contains a kinetic energy term, namely $l^2/(2mr^2) = \frac{1}{2}mr^2\dot{\theta}^2$.

It is instructive to draw an energy diagram in terms of the effective potential energy. Note that $V_{eff}$ is the sum of two terms, one positive and the other negative. For $r \to \infty$, the negative term in $V_{eff}$ is the dominant term because

$$\frac{1}{r} \bigg|_{r \to \infty} > \frac{1}{r^2} \bigg|_{r \to \infty}. $$

As $r \to 0$, the positive term dominates because

$$\frac{1}{r} \bigg|_{r \to 0} < \frac{1}{r^2} \bigg|_{r \to 0}. $$

Therefore the plot of $V_{eff}$ vs $r$ must have the general shape shown in Figure 10.4. Study this figure carefully and convince yourself it is qualitatively correct, specifically that $V_{eff}$ is positive as $r \to 0$ and negative as $r \to \infty$. Notice particularly that the $l^2/2mr^2$ term bends much more sharply than $-GMm/r$. Also, keep in mind that $r$ can only take on positive values.

Figure 10.5 is also a plot of $V_{eff}(r)$ vs $r$. In this plot the effective potential energy does not have quite the right shape because I drew it to make it easy for you to appreciate various aspects of the effective potential that are hard to see on a more accurate plot, such as Figure 10.4. In Figure 10.5 you see four possible values for the total energy, denoted $E_0$, $E_1$, $E_2$, and $E_3$. Consider first a particle with energy $E_1$. From Equations (10.9) and (10.10), we can write

$$\frac{1}{2}mr^2 = E - V_{eff}. \quad (10.11)$$

The terms “effective potential energy” and “effective potential” are used interchangeably, even though a “potential” is actually potential energy per unit mass. Some books use the term “fictitious potential.” An older term is “centrifugal potential.” In general, the effective potential is defined as the sum of $l^2/2mr^2$ and the potential energy. Thus for an electron orbiting about a proton, the effective potential would be

$$V_{eff} = \frac{l^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}. $$

At this time you may wish to review the material on energy diagrams in Section 5.6.
10.5. ENERGY AND THE EFFECTIVE POTENTIAL ENERGY

Since $\frac{1}{2}mr^2$ can never be negative, the particle cannot be located at values of $r$ for which $E < V_{\text{eff}}$. From Figure (10.5) this implies that if the energy is $E_1$, then $r \geq r_1$ where $r_1$ is the point of closest approach. You can imagine the particle starting at $r = \infty$, coming closer and closer to the primary until it reaches $r_1$, (the turning point), and then moving back out to infinity.

Figure 10.4. The effective potential energy $V_{\text{eff}}(r)$ is the sum of two terms, one positive and one negative.

Figure 10.5. Energy diagram for the effective potential. The turning points for various values of total energy are indicated.

You should keep in mind that this not a complete description of the motion; it is only a description of the radial motion. Meanwhile the particle is also moving in $\theta$ with a velocity given by $\dot{\theta} = l/mr^2$. The
angular velocity increases as the radial distance decreases, in agreement with Kepler’s second law. As we shall see, for positive values of the energy, the path of the particle is a hyperbola as illustrated in Figure 10.6. This could be the path of a comet coming in from infinity, speeding up as it approaches the Sun, swooping around the Sun, and then moving out to infinity. (Here “infinity” is a place far from the Sun, such as the Oort cloud from which many comets are believed to originate.)

\[ \dot{r} = \sqrt{\frac{2}{m}[E - V_{eff}(r)]}. \quad (10.12) \]

Thus, the square of the radial speed is proportional to \( E - V_{eff} \). (Note that \( E - V_{eff} \) is not the kinetic energy, only the radial part of the total kinetic energy.) On the energy diagram (Figure 10.5) the distance between the horizontal line at \( E_1 \) and the heavy line representing \( V_{eff} \) is proportional to \( \dot{r}^2 \). At \( r_1 \) the value of \( V_{eff} \) is \( E_1 \), so \( \dot{r} = 0 \). That is, at the turning point the particle has zero radial velocity, which makes perfect sense. At the turning point, the angular component of the velocity is a maximum because \( r\dot{\theta} = (r)(l/mr^2) = l/mr \) is greatest when \( r \) is smallest.

Next consider a particle with zero total energy \( (E = E_0 = 0; \text{ see Figure 10.5}) \). This means that the positive “radial kinetic energy” \( \frac{1}{2}m\dot{r}^2 \) is equal in magnitude to the negative effective potential energy \( V_{eff} \). The motion of the particle as it comes from \( r = \infty \) to the turning point at \( r_0 \) and then goes back out to \( r = \infty \) is similar to the motion of the particle with energy \( E_1 \). As we shall see, the main difference is that the trajectory for energy \( E_1 > 0 \) is a hyperbola and the trajectory for energy \( E_0 = 0 \) is a parabola. For the parabolic orbit, the radial speed of the particle \( \dot{r} \) is zero at infinity as well as at the turning point. As the particle comes in from infinity, \( \dot{r} \) increases, reaching a maximum at \( r_4 \) where \( V_{eff} \) reaches is greatest negative value, then slows down to
zero at \( r_0 \). The angular velocity is given by \( \dot{\theta} = l/mr^2 \). (You should be able to describe the angular velocity as the particle comes in from infinity to the point of closest approach and then moves back out to infinity.)

If the particle has negative total energy, as indicated by \( E_2 \) in Figure 10.5, the motion is quite different; there are now two turning points and the particle can neither reach \( r = 0 \) nor move out to \( r = \infty \). That is, the motion is bounded. The particle is trapped in a potential well. As it moves back and forth radially between the two points denoted by \( r_2 \) and \( r_3 \), it is also moving azimuthally with a varying angular velocity \( \dot{\theta} \). As we shall see shortly, this combination of radial and angular motion represents a trajectory which is an ellipse.

Finally, if the particle has energy \( E_3 \), the minimum possible total energy, the value of \( \dot{r} \) is zero at all times and the particle is at a constant radial position \( r = r_4 \). The path is a circle. The angular velocity is \( \dot{\theta} = l/mr^2_4 \) = constant. The particle is therefore moving with constant angular velocity in a circular path.

Depending on the value of the energy, the trajectory or orbit of the particle is a hyperbola, a parabola, an ellipse, or a circle. These are called conic sections because they can be generated by cutting a cone in various different ways, as shown in Figure 10.7.

We have been considering the motion of massive bodies interacting gravitationally, but the ideas and methods developed here are quite general and can be easily adapted to any central force problem, such as the motion of an electron in orbit around a proton in the Bohr model of the hydrogen atom.

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**Exercise 10.7.** A particle is in a parabolic orbit. Where is its turning point? Answer: \( r_0 = l^2/GMm^2 \).

**Exercise 10.8.** In the Bohr model of the hydrogen atom, an electron in the lowest orbit has angular momentum \( l = \hbar \), where \( \hbar \) is Planck’s constant divided by \( 2\pi \). The electrostatic potential energy of the electron and proton is \( V = -e^2/4\pi\varepsilon_0 r \). Show that the total energy (mechanical plus electrostatic) of the lowest orbit is \(- (1/2)me^4/(4\pi\varepsilon_0\hbar)^2\). Assume a circular orbit.
10.6. Solving the Radial Equation of Motion

The radial equation of motion (10.5) is a second order ordinary differential equation of the form

\[ \frac{d^2r}{dt^2} = \frac{l^2}{m^2r^3} - \frac{GM}{r^2}. \]

In previous chapters we learned two different ways of solving such an equation. In Section 3.6 in considering forces as a function of position, we integrated directly by setting \( \frac{d^2r}{dt^2} = v \frac{dv}{dr} \). In Section 5.7 we solved for the motion by using the energy integral in the form of Equation (5.15), namely,

\[ \int_{x_0}^{x(t)} \frac{dx}{\sqrt{E - V(x)}} = \sqrt{\frac{2}{m} t}. \]

Applying the energy approach to the present problem, we see that Equation (10.9) can be written as

\[ \frac{dr}{dt} = \left[ \frac{2}{m} \left( E - \frac{l^2}{2mr^2} + \frac{GMr}{r} \right) \right]^{\frac{1}{2}}. \quad (10.13) \]

This equation yields the following definite integral

\[ \int_{r_0}^{r(t)} \frac{dr}{\sqrt{\frac{2}{m} \left( E - \frac{l^2}{2mr^2} + \frac{GMr}{r} \right)}} = \int_0^t dt = t. \]
Integrating gives
\[ t = t(r, r_0, E, l), \]
which can, in principle, be inverted to yield
\[ r = r(t, r_0, E, l). \]
In this solution, the total energy \( E \) and angular momentum \( l \) are arbitrary constants. Note the use of the term *arbitrary constant*. Such constants are arbitrary in the sense that the differential equation is satisfied by the solution no matter what values the constants happen to have. But for a particular problem, these constants are anything but arbitrary! In simple kinematics problems the “arbitrary constants” are usually the initial position and initial velocity. In more complex problems such as the one we just solved, they tend to be other constants of the motion such as the energy and angular momentum.

**Exercise 10.9.** Obtain \( t = t(r, r_0, E, l) \) for a parabolic orbit (for which \( E = 0 \)). (You will have to look up the integral.) Note that it would be very tedious to invert the expression to obtain \( r = r(t) \). Answer:

\[
\begin{align*}
    t &= \frac{\sqrt{2m}}{3(GMm)^2} \left( G M m r + \frac{l^2}{m} \right) \sqrt{G M m r - \frac{l^2}{2m}} \\
    &\quad - \frac{\sqrt{2m}}{3(GMm)^2} \left( G M m r_0 + \frac{l^2}{m} \right) \sqrt{G M m r_0 - \frac{l^2}{2m}}
\end{align*}
\]

10.7. The Equation of the Orbit

Many physicists work in the space program. In a few years you may be involved in planning a Moon shot or a planetary probe to study the clouds of Titan. If that happens, you will need to know the position of the spacecraft as a function of time. You will have to calculate \( r = r(t) \) and you could use the technique described in the previous section.\(^8\)

On the other hand, if you wish to describe the orbit of a satellite, planet or comet, you are not interested in its position at a particular time. Rather, you want a description of the path followed by the celestial object. Mathematically, this is given by the *equation of the orbit*,

\(^8\)There are other ways to determine the position of a space probe. These are described in engineering books on “astronautics”.
$r = r(\theta)$. Once you have this equation, you can determine $r$ for every value of $\theta$ and thus trace out the path.

I will first present a “brute force” method and then a “sophisticated technique” for obtaining the orbit.

**Brute Force Method**

To determine the equation of the orbit, $r = r(\theta)$, start with the expression for $\frac{dr}{dt}$ obtained using the energy equation (Equation 10.13). Using the chain rule,

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \frac{dr}{d\theta} \dot{\theta}.$$  

But recall that the angular momentum is $l = mr^2 \dot{\theta}$ so

$$\frac{dr}{dt} = \frac{l}{mr^2} \frac{dr}{d\theta}. \quad (10.14)$$

Substituting into Equation (10.13) yields

$$\frac{dr}{d\theta} = \frac{mr^2}{l} \left[ \frac{2}{m} \left( E - \frac{l^2}{2mr^2} + \frac{GMm}{r} \right) \right]^{\frac{1}{2}}$$

$$= \left[ \frac{2mE}{l^2} r^4 - r^2 + \frac{2GMm^2}{l^2} r^3 \right]^{\frac{1}{2}}$$

$$= \left[ \alpha r^2 + \beta r^3 + \gamma r^4 \right]^{\frac{1}{2}},$$

where I used the Greek letters $\alpha$, $\beta$, and $\gamma$ to illustrate the form of this equation. Rewriting and integrating the last equation gives

$$\int_{r_0}^{r} \frac{dr}{r(\alpha + \beta r + \gamma r^2)^{1/2}} = \int_{\theta_0}^{\theta} d\theta. \quad (10.15)$$

This integral can be found in tables of integrals.

Thus, in principle, the problem of determining the equation of the orbit is solved. The only things left to do are: (1) Integrate Equation (10.15), and (2) Invert the result to obtain $r = r(\theta)$.

**Sophisticated Technique**

There is a different way to obtain the equation of the orbit that cleverly avoids evaluating the complicated integral in (10.15) and then carrying out the inversion to obtain $r = r(\theta)$. Bear with me for a little while because the math is a bit involved (but not difficult).

We will begin with Equation (10.2), the radial equation of motion:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}.$$
Using $l = mr^2\dot{\theta}$ to eliminate $\dot{\theta}$ gives Equation (10.8) which is repeated here for convenience:

$$\ddot{r} - \frac{l^2}{m^2 r^3} = \frac{-GM}{r^2}.$$ 

Now let us introduce a new variable, $u$, defined as the inverse of $r$. That is,

$$u \equiv \frac{1}{r}.$$

Then $r = u^{-1}$ and $dr = -(1/u^2)du$. Therefore,

$$\frac{dr}{dt} = -\frac{1}{u^2} \frac{du}{dt} = -\frac{1}{u^2} \frac{d\theta}{dt},$$

where the last step uses the chain rule and the fact that $u = u(\theta)$.

So,

$$\frac{dr}{dt} = -\frac{1}{u^2} \frac{d\theta}{dt} = -r^2 \frac{d\theta}{dt}.$$

But $r^2 \dot{\theta} = \frac{l}{m}$. Replacing,

$$\frac{dr}{dt} = -\frac{l}{m} \frac{du}{d\theta}.$$

Taking the derivative with respect to time again,

$$\frac{d}{dt} \left( \frac{dr}{dt} \right) = \frac{d}{dt} \left( -\frac{l}{m} \frac{du}{d\theta} \right),$$

$$\ddot{r} = -\frac{l}{m} \frac{d}{dt} \left( \frac{du}{d\theta} \right) = -\frac{l}{m} \frac{d}{d\theta} \left( \frac{du}{d\theta} \right) \frac{d\theta}{dt},$$

$$= -\frac{l}{m} \frac{d^2 u}{d\theta^2} \left( \frac{l}{mr^2} \right) = -\frac{l^2}{m^2 r^2} \frac{d^2 u}{d\theta^2}.$$

But $1/r^2 = u^2$ so

$$\ddot{r} = -\frac{l^2 u^2}{m^2} \frac{d^2 u}{d\theta^2}.$$

Substituting this into Equation (10.8) and using $1/r^3 = u^3$ we have

$$-\frac{l^2 u^2}{m^2} \frac{d^2 u}{d\theta^2} = \frac{l^2}{m^2} u^3 = -GMu^2,$$

or

$$\frac{d^2 u}{d\theta^2} + u = \frac{GMm^2}{l^2}. \quad (10.16)$$

This is a very interesting equation. It looks like the equation for simple harmonic motion except for the additional constant term on the right.
hand side. It can be made to look *exactly* like the simple harmonic motion equation by defining a new variable

\[ w = u - \frac{GMm^2}{l^2}. \]

Then, since the last term is a constant,

\[ \frac{dw}{d\theta} = \frac{du}{d\theta}, \]

and

\[ \frac{d^2u}{d\theta^2} = \frac{d^2w}{d\theta^2}. \]

Equation (10.16) can now be written in the form of the SHM equation:

\[ \frac{d^2w}{d\theta^2} + w = 0. \]

As noted in Worked Example (3.5), the solution is sinusoidal and we can write

\[ w = A \cos(\theta - \theta_0), \]

where \( A \) and \( \theta_0 \) are integration constants. Consequently,

\[ u - \frac{GMm^2}{l^2} = A \cos(\theta - \theta_0). \]

Now \( u \) is just the inverse of \( r \), so the derivation finally gives an equation for \( r \) in terms of \( \theta \):

\[ r = \frac{1}{u} = \frac{1}{\frac{GMm^2}{l^2} + A \cos(\theta - \theta_0)}. \]

Dividing top and bottom by \( GMm^2/l^2 \) puts this in a nicer form:

\[ r = \frac{\frac{l^2}{GMm^2}}{1 + \frac{AI^2}{GMm^2} \cos(\theta - \theta_0)}. \tag{10.17} \]

From a mathematical point of view, the problem is now solved because \( r \) has been expressed in terms of \( \theta \) and other known quantities (such as the angular momentum) and two constants of integration, \( A \) and \( \theta_0 \).

Equation (10.17) describes all the possible orbits for the two-body problem, i.e., circles, ellipses, parabolas, and hyperbolas. Since elliptical motion is of particular interest, let us assume that the total energy \( E \) is negative. Then the motion is bounded and the particle (planet) oscillates radially between turning points \( r_2 \) and \( r_3 \) as illustrated in Figure 10.5 for energy \( E_2 \).

Equation (10.17) contains two constants of integration, \( A \) and \( \theta_0 \). We now consider their physical significance, beginning with \( A \). At
the turning points, the effective potential energy is equal to the total energy. \( V_{\text{eff}}^{TP} = E \). This fact allows us to evaluate \( r_2 \) and \( r_3 \) which we write, in general, as \( r_{tp} \). From Equation (10.10) we have

\[
V_{\text{eff}}^{TP} = \frac{l^2}{2m r_{tp}^2} - \frac{G M m}{r_{tp}} = E.
\]

Therefore, in terms of \( u \),

\[
\frac{l^2}{2m} u_{tp}^2 - G M m u_{tp} - E = 0.
\]

The two solutions of this quadratic equation for \( u_{tp} \) are:

\[
u_\pm = \frac{m}{l^2} \left( G M m \pm \sqrt{(G M m)^2 + \frac{4l^2 E}{m}} \right).
\] (10.18)

(Note that \( 1/u_\pm \) are the \( r_2 \) and \( r_3 \) turning points of Figure 10.5.)

But we have seen that

\[
u = A \cos(\theta - \theta_0) + \frac{G M m^2}{l^2}.
\]

The maximum and minimum values of this expression occur when \( \cos(\theta - \theta_0) = \pm 1 \). That is,

\[
u_+ = A + \frac{G M m^2}{l^2},
\]

and

\[
u_- = -A + \frac{G M m^2}{l^2}.
\]

Using \( u_+ \) and equating the expression above to the relation given by Equation (10.18) leads to

\[
A + \frac{G M m^2}{l^2} = \frac{G M m^2}{l^2} + \frac{m}{l^2} \sqrt{(G M m)^2 + \frac{2l^2 E}{m}},
\]

or

\[
A = \left[ \frac{(G M m)^2 m^2}{l^4} + \frac{2Em}{l^2} \right]^\frac{1}{2}.
\] (10.19)

Thus \( A \) is related to the total energy and the angular momentum in a rather complicated way.

Let us now turn our attention to \( \theta_0 \), the other constant of integration. Considering Equation (10.17) we note that when \( \cos(\theta - \theta_0) = +1 \), the denominator is maximized and \( r \) reaches its minimum value. If \( M \) is the mass of the Sun, the place where \( r \) is a minimum is called the perihelion. The \( x \)-axis is usually taken to be a line from \( M \) through the
perihelion, as shown in Figure 10.8. If $\theta$ is measured from the $x$-axis, then $\theta_0 = 0$.

On the other hand, if we measure $\theta$ from a fixed direction in space, such as the “first point in Aries” ($\Upsilon$) then $\theta_0$ is the angle between $\Upsilon$ and the $x$-axis and gives the orientation of the ellipse relative to the fixed direction.

![Figure 10.8](image)

**Figure 10.8.** The position of a satellite relative to the major axis is given by the angle $\theta$. The major axis can be defined to be the $x$ axis. The angle $\theta_0$ gives the orientation of the elliptical orbit relative to a fixed line in space.

We have, then, determined $A$ in terms of $E$ and $l$ and given the physical interpretation of $\theta_0$ as a description of the orientation of the ellipse with respect to inertial axes. Now the problem is indeed fully solved. Unfortunately, as you have no doubt noticed, our expressions are rather unwieldy. In a moment I will write them in a neater form. But first we need to consider some properties of conic sections, particularly ellipses.

### 10.8. The Equation of an Ellipse

You probably remember the grade school method for drawing an ellipse. You place a sheet of paper on a cork board and stick two tacks in it. Then you take a piece of string, tie the ends together, and loop it around the tacks. Finally you place a pencil in the loop of string and move it around the tacks, keeping the string taut, and if you are very careful you will draw an ellipse on the paper. See Figure 10.9.

As the pencil traces out the ellipse, the distance from the pencil to either tack varies. But the sum of the distances to the two tacks remains the same because the string is of constant length and the distance between the two tacks does not change. Therefore the distance from the first tack to the pencil plus the distance from the pencil to the second
An ellipse is the locus of points whose distances from two fixed points sum to a constant.

That is a bit complicated to say, but it is expressed quite simply as an equation. First choose the two fixed points, call them focal points and denote them $F$ and $F'$. The distance from a point on the ellipse to $F$ will be called $r$ and the distance from that same point to $F'$ will be $r'$. See Figure 10.10. Then an ellipse is defined as the locus of points such that $r + r' = \text{constant}$.

It is easy to obtain an equation for the ellipse in polar coordinates by introducing the angle $\theta$ between $r$ and the major axis of the ellipse. The length of the semimajor axis ($\frac{1}{2}PP'$) will be denoted $a$, and the semiminor axis is denoted $b$. The distance from the center of the ellipse to either focal point will be $f = ea$, where $e$ is a number less than one, called the eccentricity. The eccentricity $e$ and the semimajor axis $a$ determine the size and shape of the ellipse.
It turns out that
\[ r + r' = 2a. \]
This relation may surprise you, but it is easily proved. Consider point \( P \). There \( r = FP \) and \( r' = F'P' \). By symmetry, \( FP = F'P' \) and therefore \( r + r' = FP + F'P' = 2a \). Since \( r + r' \) is a constant, if it is equal to \( 2a \) at one point, it is equal to \( 2a \) at any point.

Now consider the triangle formed by \( r, r', \) and \( 2f \), where \( f = ea \). See Figure 10.10. According to the law of cosines, if \( \theta \) is the “external” angle,
\[ r'^2 = r^2 + (2ae)^2 + 2r(2ae) \cos \theta. \] (10.20)
Substituting \( r' = 2a - r \) into Equation (10.20) gives
\[ (2a - r)^2 = r^2 + 4a^2e^2 + 4ae \cos \theta, \]
and a little bit of algebra leads to
\[ r = \frac{a(1 - e^2)}{1 + e \cos \theta}. \] (10.21)
This is the equation for an ellipse in plane polar coordinates. If we were to go through the same process for hyperbolas and parabolas, we would obtain similar equations for \( r = r(\theta) \). In fact, the equation for any conic section can be expressed in the form
\[ r = \frac{p}{1 + e \cos \theta}, \] (10.22)
where \( p = a(1 - e^2) \) for an ellipse, \( p = a(e^2 - 1) \) for a hyperbola\(^9\) and \( p = a \) for a parabola \((a \) is a characteristic constant for each type of curve and, in general, \( p = l^2/GMm^2 \).) The various curves correspond to different choices of the eccentricity \( e \). Specifically,
\[ e = 0 \quad \text{gives a circle}, \]
\[ e < 1 \quad \text{gives an ellipse}, \]
\[ e = 1 \quad \text{gives a parabola}, \]
\[ e > 1 \quad \text{gives a hyperbola}. \]

The semimajor axis \( a \) determines the size of an ellipse and the eccentricity determines its shape. At zero eccentricity, the ellipse degenerates into a circle. As the eccentricity gets closer and closer to one, the ellipse gets flatter and flatter. That is, as \( e \) approaches unity, the

\(^9\)A hyperbola has two branches and is described by the equation
\[ r = \frac{a(e^2 - 1)}{\pm 1 + e \cos \theta}, \]
where the “plus branch” corresponds to an attractive force and the “minus branch” corresponds to a repulsive force.
10.8. THE EQUATION OF AN ELLIPSE

ratio of the semiminor axis to the semimajor axis approaches zero. As you will prove in Exercise 10.11,

\[ \frac{b}{a} = (1 - e^2)^{1/2}. \]  (10.23)

You probably know from geometry that the area of an ellipse is given by \( S = \pi ab \). It is sometimes convenient to write this in terms of \( a \) only, thus, using Equation 10.23,

\[ S = \pi a^2 (1 - e^2)^{1/2}. \]  (10.24)

Having considered the properties of conics, let’s go back to the motion of planets. Comparing the equation of an ellipse (10.21) with the equation of the orbit (10.17), you can see that the orbit is an ellipse (or, more generally, a conic section). A term by term comparison of these two equations shows that

\[ \frac{l^2}{GMm^2} = a(1 - e^2), \]  (10.25)

and that

\[ \frac{Al^2}{GMm^2} = e. \]

But we showed in Equation (10.19) that

\[ A = \left[ \frac{(GMm)^2m^2}{l^4} + \frac{2Em}{l^2} \right]^{1/2}, \]

so

\[ e^2 = \frac{(GMm)^2m^2}{l^4} \frac{l^4}{(GMm)^2m^2} + \frac{2Em}{l^2} \frac{l^4}{(GMm)^2m^2}. \]

Therefore,

\[ e^2 = 1 + \frac{2El^2}{m(GMm)^2}. \]  (10.26)

This gives us an expression for the eccentricity in terms of constants of the motion. Note that for \( E > 0 \) we get \( e > 1 \), a hyperbola. For \( E = 0 \) we get \( e = 1 \), a parabola. For \( E < 0 \), we get \( e < 1 \), an ellipse. Finally, for a circle, \( e = 0 \) so \( 2El^2/m(GMm)^2 = -1 \).

Plugging Equation (10.26) into (10.25) gives the following expression for the semimajor axis:

\[ a = -\frac{GMm}{2E}. \]  (10.27)

The minus sign is necessary because the total energy is negative for an ellipse. Equation 10.27 shows that the semimajor axis depends only on the energy.
Worked Example 10.3. A comet of mass $m$ starts from infinity with velocity $v_0$ and impact parameter $b$. (If undeflected, the path of the comet would be a straight line passing the Sun at a distance $b$.) (a) Show that the distance of closest approach to the Sun is approximately $b^2v_0^2/GM$. (b) Write the equation of the orbit in polar coordinates in terms of $M, v_0$, and $b$, where $M$ is the mass of the Sun.

Solution: (a) Let the Sun-comet distance at perihelion be denoted $d$. At that point, the speed of the comet is $v_d$ and its velocity is perpendicular to the Sun-comet line. By conservation of angular momentum $l_\infty = l_d$, and

$$mv_0b = mv_d d,$$

or,

$$b/d = v_d/v_0.$$

By conservation of energy

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_d^2 - \frac{GMm}{d},$$

so dividing by $v_0^2$,

$$1 = \left(\frac{b}{d}\right)^2 - \frac{2GM}{dv_0^2}.$$

Multiply by $d^2$ and obtain the quadratic

$$d^2 + \frac{2GM}{v_0^2} d - b^2 = 0,$$

$$d = -\frac{GM}{v_0^2} \pm \sqrt{\left(\frac{GM}{v_0^2}\right)^2 + b^2}.$$

Since $d > 0$, we use the positive sign and write

$$d = \frac{GM}{v_0^2} \left(-1 + \left[1 + \frac{b^2v_0^4}{2G^2M^2}\right]^{1/2}\right).$$

Apply the binomial expansion to get

$$d = \frac{GM}{v_0^2} \left(-1 + \left[1 + \frac{b^2v_0^4}{2G^2M^2} + \cdots \right]\right),$$

$$d \approx \frac{GM}{v_0^2} \frac{b^2v_0^4}{2G^2M^2} = \frac{b^2v_0^2}{2GM}.$$
(b) In polar coordinates the hyperbolic orbit under an attractive force is

\[ r = \frac{a(e^2 - 1)}{1 + e \cos \theta}. \]

The problem asks us to express \( a \) and \( e \) in terms of the given parameters, namely \( v_0, b \) and constants. The speed of the comet squared is

\[ v^2 = r^2 + r^2 \dot{\theta}^2. \]

Taking the derivative of \( r \) we have

\[ \dot{r} = \frac{[a(e^2 - 1)]e \sin \theta \dot{\theta}}{(1 + e \cos \theta)^2}. \]

Now \( l = mr^2 \dot{\theta} \) so

\[ \dot{\theta} = \frac{l}{mr^2} = \frac{l}{m} \frac{1 + e \cos \theta}{a(e^2 - 1)^2} \]

and we can write

\[ \dot{r}^2 = \frac{e^2 r^2 / m^2}{[a(e^2 - 1)]^2} \sin^2 \theta. \]

The second term in \( v^2 \) is \( r^2 \dot{\theta}^2 \). But,

\[ r \dot{\theta} = \frac{1}{r} \frac{l}{m} = \frac{l}{m} \frac{1 + e \cos \theta}{a(e^2 - 1)}, \]

so

\[ v^2 = \frac{l^2}{m^2} \frac{(1 + 2e \cos \theta + e^2 (\sin^2 \theta + \cos^2 \theta))}{[a(e^2 - 1)]^2}. \]

\[ = \frac{l^2}{m^2 a(e^2 - 1)} \left( \frac{2 + 2e \cos \theta + e^2 - 1}{a(e^2 - 1)} \right) \]

\[ = \frac{l^2}{m^2 a(e^2 - 1)} \left( \frac{1 + e \cos \theta + e^2 - 1}{a(e^2 - 1)} \right) \]

\[ = \frac{l^2}{m^2 a(e^2 - 1)} \left[ \frac{2}{r} + \frac{1}{a} \right]. \]

For an elliptical orbit, \( r = \frac{a(1-e^2)}{1+e \cos \theta} \) and \( l^2 = GMm^2a(1-e^2) \).

Similarly, for a parabolic orbit \( r = \frac{a(e^2-1)}{1+e \cos \theta} \), so \( l^2 = GMm^2a(e^2-1) \).
Consequently

\[ v^2 = \frac{GMm^2a(e^2 - 1)}{m^2a(e^2 - 1)} \left( \frac{2}{r} + \frac{1}{a} \right) = GM \left[ \frac{2}{r} + \frac{1}{a} \right]. \]

At \( r = \infty \), the velocity is \( v_0 \) and we see that

\[ v_0^2 = \frac{GM}{a}, \]

so

\[ a = \frac{GM}{v_0^2}. \]

Also

\[ a(e^2 - 1) = \frac{l^2}{GMm^2}. \]

Solving for \( e \) we obtain

\[ e = \left( 1 + \left[ \frac{v_0b}{(GM)} \right]^2 \right)^{1/2}, \]

and finally

\[ r = \frac{v_0^2b^2/GM}{1 + \sqrt{1 + \left( \frac{b_0^2}{GM} \right)^2 \cos \theta}}. \]

**Exercise 10.10.** Starting with Equation (10.20), obtain Equation (10.21).

**Exercise 10.11.** Show that for an ellipse \( b/a = (1 - e^2)^{1/2} \). (Hint: Apply the Pythagorean theorem to the triangle FOP in Figure 10.11.)

**Figure 10.11.** An ellipse. Note that the distance from the center to the focal point is \( ea \) and that the sum of the distances \( FP \) and \( F'P \) is \( 2a \).

**Exercise 10.12.** A planet is in an elliptical orbit with semimajor axis \( a \). By averaging the largest and smallest values of \( r \), show that it
leads to an “average” value of the potential energy equal to $-\frac{GMm}{a}$.
(Compare with the virial theorem discussed in Worked Example 10.2.)

**Exercise 10.13.** Plot Equation (10.22) for $e = 2, e = 1.0, e = 0.5,$
and $e = 0.$
Figure 10.12. Kepler’s second law states that equal areas are swept out in equal times.

We said, “For a central force, the angular momentum is constant.” Are the two statements equivalent? Does the fact that the angular momentum is constant imply that the areal velocity is constant? The answer is yes. To prove that this is true, consider a planet located at vector position \( \mathbf{r} \). After a time interval \( dt \) the planet will be at \( \mathbf{r} + d\mathbf{r} = \mathbf{r} + \mathbf{v}dt \). See Figure 10.13. The shaded region is the area swept out. Recall that \( |\mathbf{a} \times \mathbf{b}| \) is equal to the area of a parallelogram whose sides are the vectors \( \mathbf{a} \) and \( \mathbf{b} \). The shaded region in Figure 10.13 is one half of the parallelogram formed from \( \mathbf{r} \) and \( d\mathbf{r} \). Therefore, the area swept out by the planet’s radius vector in time \( dt \) is

\[
dS = \frac{1}{2} |\mathbf{r} \times \mathbf{v}dt| = \frac{dt}{2} |\mathbf{r} \times \mathbf{v}|.
\]

But by the definition of angular momentum, \( |\mathbf{r} \times \mathbf{v}| = l/m \). Therefore,

\[
dS = \frac{1}{2m} \frac{l}{dt}dt,
\]

or

\[
\frac{dS}{dt} = \text{areal velocity} = \frac{l}{2m}.
\]

Figure 10.13. The area of the triangle is one half the area of the parallelogram. Therefore,

\[
\Delta S = \frac{1}{2} |\mathbf{r} \times d\mathbf{r}| = \frac{1}{2} |\mathbf{r} \times \mathbf{v} \Delta t|.
\]

The mass of a planet is constant, so the areal velocity is proportional to the angular momentum and \( \frac{dS}{dt} = \text{constant} \) is completely equivalent.
10.9. KEPLER’S LAWS REVISITED

Kepler’s second law is simply a consequence of the fact that in central force motion, the angular momentum is conserved.

Kepler’s third law states:

**The period of a planet squared is proportional to its semimajor axis cubed.**

In mathematical terms, if $\tau$ is the time required for a planet to orbit the Sun and if $a$ is the semimajor axis of the orbit, the third law can be written

$$\tau^2 = Ka^3,$$

where $K$ is some constant. Kepler discovered this law after years of studying planetary data. On the other hand, our analytical study of elliptical motion leads to it in a fairly simple way. We just note that

$$\frac{dS}{dt} = \frac{l}{2m} \Rightarrow \oint_{\text{orbit}} dS = \frac{l}{2m} \oint_{\text{orbit}} dt.$$

Let $S$ equal the area of the ellipse. Then if $\tau$ is the period,

$$S = \frac{l}{2m} \tau.$$

But we have seen that $S = \pi a^2(1 - e^2)^{1/2}$ so

$$\frac{l}{2m} \tau = \pi a^2(1 - e^2)^{1/2}.$$

According to Equation (10.25), $(1 - e^2) = \frac{l^2}{GMm^2a}$. Squaring we obtain

$$\frac{l^2}{4m^2} \tau^2 = \pi^2 a^4 \frac{l^2}{GMm^2a}$$

or

$$\tau^2 = \frac{4\pi^2}{GM} a^3.$$

This relation states that the period squared is proportional to the semimajor axis cubed. Thus, Kepler’s third law is proved. Note that the constant of proportionality depends only on the mass of the Sun, so the ratio $\tau^2/a^3$ should be the same for all the planets. Therefore, determining the period of a planet or asteroid allows us to evaluate its semimajor axis. Actually, this result is not completely accurate. There is a very small correction that is due to the fact that the Sun is not really at rest at the origin. The Sun and the planet move around their common center of mass. This point is almost, but not quite, at the center of the Sun. This introduces a small correction into Kepler’s third law so that the quantity $M$ in the denominator should be replaced by $M + m$. Once again we appreciate the technique in physics.
of first solving an easier idealized problem and introducing the complications later. (Problem 10.9 asks you to carry out the appropriate calculations.)

**Worked Example 10.4.** A particle moves in an elliptical orbit with semimajor axis $a$ and eccentricity $e$. The velocity of the particle is observed to vary from a minimum value $v_1$ to a maximum value $v_2$. Determine the period.

**Solution:** The angular momentum of the particle is constant:

$$l = mvr = \text{constant}.$$ 

Therefore, the maximum (and minimum) speeds occur when $r$ is at the smallest (and greatest) distance from the force center. Call these distances $r_1$ and $r_2$. Note that

$$mv_1 r_1 = mv_2 r_2.$$ 

Furthermore, it is easy to appreciate that the points at $r_1$ and $r_2$ as well as the force center all lie on the semimajor axis. Consequently,

$$r_1 + r_2 = 2a.$$ 

Hence

$$l = mv_1 (2a - r_2) = mv_2 r_2,$$

$$r_2 = \frac{2av_1}{v_1 + v_2},$$

so the angular momentum can be written

$$l = m \frac{2av_1 v_2}{v_1 + v_2}.$$ 

The angular momentum is related to the areal velocity by

$$\frac{l}{2m} = \frac{dS}{dt} = \text{areal velocity}.$$ 

The period is equal to the area divided by the areal velocity

$$\tau = \frac{\text{area}}{\text{areal velocity}} = \frac{\pi ab}{(l/2m)} = \frac{\pi ab(2m)}{m \frac{2av_1 v_2}{v_1 + v_2}} = \frac{\pi b(v_1 + v_2)}{v_1 v_2}.$$ 

But $b$ is the semiminor axis and equal to $a(1 - e^2)^{1/2}$, so

$$\tau = \pi a(1 - e^2)^{1/2} \frac{v_1 + v_2}{v_1 v_2}.$$
**Worked Example 10.5.** Suppose you were familiar with Kepler’s work, so you knew that the orbits of planets are ellipses and that the angular momentum is constant. This suggests to you that the force on the planet is a central force, so $F = f(r)$, but you don’t know any other properties of the force. Show that the force obeys the inverse square law, that is, show that $f(r) \propto 1/r^2$.

**Solution.** The radial equation of motion of motion Equation (10.2) can be written

$$\ddot{r} - r\dot{\theta}^2 = -f(r)/m,$$

and the angular equation of motion leads to

$$\dot{\theta} = l/mr^2,$$

so

$$\ddot{r} - \frac{l^2}{m^2 r^3} = -\frac{f(r)}{m}.$$  

In terms of $u = 1/r$, recalling that $\ddot{r} = -\frac{l^2 u^2}{m^2} \frac{d^2 u}{d\theta^2}$, we obtain

$$\frac{l^2 u^2}{m^2} \frac{d^2 u}{d\theta^2} - \frac{l^2}{m^2 u^3} = -\frac{f(r)}{m},$$

$$\frac{d^2 u}{d\theta^2} + u = \frac{m}{l^2 u^2} f(r). \quad (10.28)$$

Recall that the equation of an ellipse has the form

$$r = \frac{p}{1 + e \cos \theta},$$

\[ \therefore \quad u = \frac{1}{p} + \frac{e}{p} \cos \theta. \]

Therefore

$$\frac{d^2 u}{d\theta^2} = -\frac{e}{p} \cos \theta,$$

and Equation (10.28) becomes

$$-\frac{e}{p} \cos \theta + \left( \frac{1}{p} + \frac{e}{p} \cos \theta \right) = \frac{m}{l^2 u^2} f(r)$$

$$\frac{1}{p} = \frac{m}{l^2 u^2} f(r) = \frac{m}{l^2} \frac{r^2}{r^2} f(r).$$

That is,

$$f(r) = \frac{l^2}{mp} \frac{1}{r^2} \propto \frac{1}{r^2}$$

Q.E.D.
Exercise 10.14. A certain asteroid in the solar system has a period of four years. What is its semimajor axis in AU? (Assume you do not know the gravitational constant $G$ or the mass of the Sun, but you do know that the Earth is at a distance of 1 AU from the Sun.) Answer: 2.52 AU.

Exercise 10.15. (a) Using the form $\tau^2 = \left(\frac{4\pi^2}{GM}\right)a^3$ and assuming you do not know the mass of the Sun, determine the semimajor axis of Saturn in AU, given its period is 29.5 Earth years. (b) Using the correction $(M + m)$ and looking up the appropriate values, obtain a corrected value. Answers: 9.55 AU, 9.52 AU

Exercise 10.16. For a planet in a circular orbit, $F = ma$ can be written as $GMm/r^2 = mv^2/r$. Use this to derive Kepler’s third law for circular orbits.

Exercise 10.17. Halley’s comet has an eccentricity of 0.967. Its perihelion distance is $8.81 \times 10^{10}$ m. What is its period? Answer: 75.4 years.

10.10. A Perturbed Circular Orbit

In this section we will study the *stability* of a circular orbit. When a physical system in an equilibrium state is perturbed by a small force, it will often oscillate about equilibrium. This is considered a *stable* equilibrium. On the other hand, if the perturbation causes the system to undergo a large change, the equilibrium is unstable. (The usual example of stable equilibrium is a marble at the bottom of a depression, and unstable equilibrium would be a marble balanced on the peak of a hill.) This section is both an application of the concepts studied in the previous sections and an introduction to the methods for dealing with small perturbations.

Consider a particle moving in a circular orbit under the action of a central force. For reasons that will eventually become clear, I will not require the force to obey the inverse square law. That is, although $\mathbf{F} = f(r)\mathbf{r}$, the functional form of $f(r)$ will be left undetermined for now.

The equations of motion are given by generalizations of Equations (10.2):
10.10. A PERTURBED CIRCULAR ORBIT

\[ m(\ddot{r} - r\dot{\theta}^2) = f(r), \quad (10.29) \]
\[ m(r\ddot{\theta} + 2r\dot{\theta}) = 0. \]

As before,
\[ \dot{\theta} = \frac{l}{mr^2}, \quad (10.30) \]
with \( l = \text{constant} \). Inserting this expression for \( \dot{\theta} \) into the radial equation of motion (Equation 10.29), leads to
\[ m\ddot{r} = f(r) + \frac{l^2}{mr^3}. \quad (10.31) \]

If the particle is moving in a circular orbit of radius \( a \), then
\[ r = a = \text{constant}. \]

Consequently,
\[ \ddot{r} = 0. \]

The radial equation of motion (Equation 10.31) then reduces to
\[ f(a) = -\frac{l^2}{ma^3}. \quad (10.32) \]

I will come back to this equation shortly.

Now consider the question of the stability of a perturbed circular orbit. For example, we might be considering the motion of a planet in a perfectly circular orbit which is hit by a comet (as happened some years ago when comet Shoemaker-Levy collided with Jupiter). We want to know if the planet will continue moving in a stable orbit after a slight perturbation, or if it will behave in an erratic manner. In other words, we would like to know if a collision with a relatively small body could cause Jupiter to go flying out of the solar system.

After a collision or some other sort of perturbation, the radial position is no longer exactly equal to \( a \), but it is still nearly equal to \( a \), so we can write
\[ r = a + \eta, \quad (10.33) \]
where \( \eta \ll a \), that is, \( \eta \) is a very small quantity compared to \( a \).

Inserting expression (10.33) for \( r \) into the radial equation of motion in the form (10.31) we obtain
\[ m\frac{d^2}{dt^2}(a + \eta) = f(a + \eta) + \frac{l^2}{m(a + \eta)^3}. \]
or
\[ m\ddot{\eta} = f(a + \eta) + \frac{l^2}{ma^3(1 + \frac{\eta}{a})^3}. \]  

(10.34)

The force at the position \( r = a + \eta \) is very nearly equal to the force at \( r = a \) so we are justified in expanding \( f(a + \eta) \) in a Taylor’s series expansion and keeping only the leading terms. Thus:
\[ f(a + \eta) = f(a) + \eta \frac{df}{dr}_{r=a} + \frac{1}{2} \eta^2 \frac{d^2f}{dr^2}_{r=a} + \cdots. \]

The term \( (1 + \eta/a)^3 \) in the denominator of the last term of Equation (10.34) can also be expanded. Using the binomial expansion:
\[ \frac{1}{(1 + \eta/a)^3} = \left( 1 + \frac{\eta}{a} \right)^{-3} = 1 - 3\frac{\eta}{a} + 6\left( \frac{\eta}{a} \right)^2 + \cdots. \]

Consequently, Equation (10.34) can be written (to first order in \( \eta \)) as
\[ m\ddot{\eta} = f(a) + \eta \frac{df}{dr}_{r=a} + \cdots + \frac{l^2}{ma^3}(1 - 3\frac{\eta}{a} + \cdots), \]
or
\[ m\ddot{\eta} = f(a) + \eta \frac{df}{dr}_{r=a} + \frac{l^2}{ma^3} - 3\frac{\eta}{a} \frac{l^2}{ma^3}. \]

Now recall that Equation (10.32) states that \( f(a) = -\frac{l^2}{ma^3} \), so the first and third terms on the right hand side cancel. This leaves
\[ m\ddot{\eta} \equiv \eta \frac{df}{dr}_{r=a} - \frac{l^2}{ma^3} \]
\[ = \eta \left( \frac{df}{dr}_{r=a} - \frac{3l^2}{ma^4} \right), \]
or
\[ \ddot{\eta} + \eta \left( \frac{3l^2}{m^2a^4} - \frac{1}{m} \frac{df}{dr}_{r=a} \right) = 0. \]  

(10.35)

This equation has the form
\[ \ddot{\eta} + K\eta = 0. \]

---

\[ \textit{If you have not already done so, you should immediately memorize the following extremely important series expansions:} \]

\[ \text{Taylor’s Series:} \]
\[ F(a + \delta x) = F(a) + \delta x \frac{dF}{dx}_{r=a} + \frac{1}{2!} \delta x^2 \frac{d^2F}{dx^2}_{r=a} + \frac{1}{3!} \delta x^3 \frac{d^3F}{dx^3}_{r=a} + \cdots. \]

\[ \text{Binomial Expansion:} \]
\[ (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots. \]
We will be considering equations of this form in detail later, but for now it is sufficient to note that the general form of the solution depends on whether $K$ is positive or negative. For positive $K$ the solution has the form

$$\eta = A \sin \sqrt{K} t + B \cos \sqrt{K} t,$$

and for negative $K$ the solution is

$$\eta = C e^{+\sqrt{|K|} t} + D e^{-\sqrt{|K|} t}.$$

Thus, for positive $K$ the solution is a sinusoidal oscillation, as in simple harmonic motion (see Section 3.6). On the other hand, if $K$ is negative, the solution is the sum of an exponential decrease plus an exponential increase. The exponentially decreasing term quickly dies out and $\eta(t)$ increases with time exponentially. We can interpret this result physically to mean that for $K < 0$ the orbit is unstable because the “distance” $\eta$ from the radius $a$ of a circular orbit grows without bounds. On the other hand, if $K > 0$, then $\eta$ oscillates back and forth around zero, meaning that the particle oscillates about $r = a$ in simple harmonic motion. Therefore, the condition for a stable orbit is that $K > 0$. That is,

$$\frac{3l^2}{m^2 a^4} - \frac{1}{m} \left. \frac{df}{dr} \right|_{r=a} > 0.$$

Once again we can use the fact that $l^2/ma^3 = -f(a)$, and rewrite this equation as

$$-\frac{3}{a} f(a) - \left. \frac{df}{dr} \right|_{r=a} > 0. \quad (10.38)$$

The gravitational force is $f = -GMm/r^2$ and $\left. \frac{df}{dr} \right|_{r=a} = +2GMm/a^3$. So, for gravity, Equation (10.38) becomes

$$\frac{3}{a} \left( \frac{GMm}{a^2} \right) - \frac{2GMm}{a^3} = \frac{GMm}{a^3} > 0,$$

and consequently, the gravitational force leads to stable orbits.

But what if the force is not inverse square? A more general central force might have the form

$$f = -\frac{c}{r^n}.$$  

Here $n$ is an integer that can be greater or less than zero. (For the gravitational force, $n = 2$; notice that I explicitly incorporated the negative sign on the right hand side because only attractive force laws lead to orbital motion.)
If \( f = -\frac{c}{r^n} \), then

\[
\frac{df}{dr} \bigg|_a = \frac{cn}{a^{n+1}}.
\]

Consequently, the stability condition (10.38) is

\[
\frac{3}{a} \left( -\frac{c}{r^n} \right) \bigg|_a - \frac{cn}{a^{n+1}} > 0
\]

\[
\frac{3}{a} \left( -\frac{c}{a^n} \right) - \frac{cn}{a^{n+1}} > 0
\]

\[
\frac{c}{a^{n+1}} (3 - n) > 0
\]

or

\[
n - 3 < 0
\]

The inverse square force law \( (n = 2) \) leads to orbits that are stable under small perturbations. If the gravitational force had the form \( F \propto \frac{1}{r^4} \), a planetary system would be impossible because any tiny perturbation would cause the planets to spiral away from their orbits.

Notice that in determining the stability of circular orbits, we also solved the problem of determining the frequency of small oscillations in a perturbed orbit because Equation (10.35), the equation of motion for \( \eta \), has the form

\[
\ddot{\eta} + K \eta = 0,
\]

and, as mentioned previously, this is the equation for simple harmonic motion. The frequency of harmonic motion is \( \omega_0 = \sqrt{K} \), so

\[
\omega_0^2 = \frac{3l^2}{m^2a^4} - \frac{1}{m} \left. \frac{df}{dr} \right|_{r=a}
\]

\[
= - \frac{3}{ma} f(a) - \frac{1}{m} \left. \frac{df}{dr} \right|_{r=a}.
\]

For a particle subjected to an inverse square force law, such as the gravitational force,

\[
f = -\frac{c}{r^2},
\]

\[
\left. \frac{df}{dr} \right|_{r=a} = +2cr^{-3} = 2c \frac{a^3}{a^3} = \frac{2c}{a^3}.
\]

Also

\[
f(a) = -\frac{c}{a^2}.
\]
Consequently,

$$\omega_0^2 = -\frac{3}{ma} \left( -\frac{c}{a^2} \right) - \frac{1}{m} \left( \frac{2c}{a^3} \right) = \frac{c}{ma^3}. \quad (10.39)$$

This equation gives the frequency of the radial oscillations. Note that $2\pi/\omega_0$ is the time for the planet to go from maximum $\eta$ to minimum $\eta$ and back to maximum $\eta$. While carrying out this radial motion, the planet has an angular velocity given by\(^{11}\)

$$\dot{\theta} = \frac{l}{mr^2} \bigg|_{r=a} = \frac{l}{ma^2}.$$ 

Therefore,

$$\dot{\theta}^2 = \frac{l^2}{m^2a^4} = \frac{-f(a)ma^3}{m^2a^4} = -\frac{(c/a^2)ma^3}{m^2a^4} = \frac{c}{ma^3}. \quad (10.40)$$

Comparing Equation (10.39) with Equation (10.40) we see that $\omega_0 = \dot{\theta}$, so for a $1/r^2$ force law, the radial oscillations have exactly the same frequency as the orbital motion. Therefore, in the time required for the particle to go completely around in its orbit, it will go through one complete radial oscillation. The combination of these two motions converts a circular orbit into an elliptical orbit.\(^{12}\)

It can happen that an orbiting particle is perturbed in such a way that the orbital period and the period of radial oscillation are not equal. For example, the Earth is not a perfect sphere; it is an oblate spheroid and bulges slightly at the equator. This bulge exerts a force on an orbiting artificial satellite, and gives rise to a radial oscillation of frequency $\omega_r$ that is not equal to $\dot{\theta}$. The satellite moves in an elliptical orbit, but since the radial motion is not exactly synchronized with the angular motion, the ellipse slowly precesses. In other words, the major axis of the ellipse slowly rotates at a rate which is the difference between the two oscillations. The angular frequency of the precession of the ellipse, $\omega_p$ is given by

$$\omega_p = \omega_r - \dot{\theta}$$

where $\omega_r$ is the frequency of the radial oscillations.

---

\(^{11}\)Actually we should write the average value of the angular velocity, that is, $<\dot{\theta}>$ rather than $\dot{\theta}$, but since $<\dot{\theta}> = \dot{\theta}$, I used $\dot{\theta}$ for simplicity in notation.

\(^{12}\)You might conclude that the eccentric orbits of planets are a result of collisions with asteroids and comets. However, astronomers do not believe that this accounts for the eccentricities of planetary orbits.
Exercise 10.18. Draw the orbit of a planet assuming the radial oscillation has a period half that of the angular motion.

Exercise 10.19. By simple substitution, show that Equations 10.36 and 10.37 are solutions to the simple harmonic motion differential equation.

Figure 10.14. Commensurable and incommensurable orbits. The two orbits on the left are commensurable and sooner or later the planet traces out the same path. The orbit on the right is incommensurable and the planet never repeats exactly the same path.

10.11. Resonances

An orbit is usually thought of as the path of a particle that retraces its motion over and over again, as in the case of a single planet around a star. If the particle always passes through the same points, the orbit is said to be closed. In a simple closed orbit, the radial period and the angular period are equal. That is, the planet goes from one turning point to the other and back again in the same time as the angular displacement goes from 0 to $2\pi$. If the radial period is slightly greater than the angular period, the planet will reach perihelion a short time after its angular position has advanced $2\pi$. The orbit is not closed and the perihelion precesses. The orbit can actually be closed even if the two periods are not equal as long as they are in the ratio of integers, as in the middle picture of Figure 10.14. When two periods or (equivalently) two frequencies are in the ratio of small integers, they are said to be commensurable or in resonance. If the ratio of the radial frequency to the angular frequency is an irrational number such as $\pi$ or $\sqrt{2}$, the orbit is not closed; it will never repeat itself. Figure 10.14 illustrates two commensurable orbits and one incommensurable orbit.
Resonances are very important in celestial mechanics. One often hears of a system being “locked” into a particular resonance. For example, the rotation of the Moon is locked into a 1:1 resonance with its orbital motion. Therefore, the same side of the Moon always faces Earth. Similarly, the orbit of a 24 hour satellite is in a 1:1 resonance with the rotation of the Earth. An interesting case is the 2:5 resonance between the periods of Jupiter and Saturn. In old fashioned terminology used in celestial mechanics, this resonance is called “The Great Inequality.”

10.12. Summary

A central force is directed towards or away from the origin and has a magnitude that depends only on the distance to the origin, \( F = f(r)\hat{r} \). An example is the gravitational force acting on a planet as it orbits the Sun.

Kepler studied the motion of the planets and determined that this motion obeys three relations which we call Kepler’s Laws.

A particle moving in a central force field has constant angular momentum. The equations of motion for a particle of mass \( m \) in the field of a body of mass \( M \) are

\[
\ddot{r} - \frac{l^2}{m^2r^3} = -\frac{GM}{r^2}
\]

\[
\frac{d}{dt}(mr^2\dot{\theta}) = \frac{dl}{dt} = 0.
\]

The effective potential energy for a particle in a gravitational field is

\[
V_{\text{eff}} = \frac{l^2}{2mr^2} - \frac{GMm}{r}.
\]

The radial motion of such a particle can be visualized on a plot of \( V_{\text{eff}} \) vs \( r \), such as Figure 10.5. Such plots indicate that the orbit of the particle (planet) is a conic section.

The equations of motion can be used to obtain the position of the planet as a function of time or to determine the equation of the orbit, \( r = r(\theta) \). If the total energy is negative, the equation of the orbit is

\[
r = \frac{l^2/GMm^2}{1 + \frac{A^2}{GM} \cos(\theta - \theta_0)},
\]

which has the form of the equation for an ellipse,

\[
r = \frac{a(1 - e^2)}{1 + e \cos \theta}.
\]
Using these relations one can relate Kepler's laws to physical concepts as follows.

First Law: Planets move in elliptical orbits. This is a consequence of the inverse square nature of the gravitational force.

Second Law: Planets sweep out equal areas in equal times. This is a consequence of the conservation of angular momentum in a central force field.

Third Law: The period squared is proportional to the semimajor axis cubed. This is a consequence of the conservation of energy and the fact that the magnitude of the total energy is inversely proportional to the semimajor axis.

When a planet in a circular orbit is perturbed, it will oscillate radially around the circular path, converting the orbit into an ellipse. The orbit is stable for any force law having the form $f = -cr^n$ as long as $n$ is greater than -3.

10.13. Problems

Problem 10.1. (Bohr model of the atom.) Consider an electron in a circular orbit around a proton. Assume the angular momentum can only take on values equal to $n\hbar$ where $n$ is an integer ($n = 1, 2, 3, ...$) and $\hbar$ is a constant. Determine the possible values of the radius for the orbit and the possible values for the total energy. Make up a table of the energy for the first four energy states (that is, for the electron in the four smallest orbits).

Problem 10.2. A particle of mass $m$ is in a Lennard-Jones force field. Assume the force center is stationary. The particle has velocity $v$.

(a) Write the Lagrangian
(b) Obtain the equations of motion
(c) Determine the angular momentum and show that it is a constant.

Problem 10.3. A bola is made up of two or three spheres about 5 cm in diameter attached to one another by a thin rope. A gaucho (an Argentine cowboy) will throw a bola to wrap around the legs of a calf, bird, or some other animal. Consider a two ball bola made up of masses $m_1$ and $m_2$ connected by an ideal string of length $b$. The gaucho holds $m_1$ in his hand and spins $m_2$ around in a circle. Suppose he releases $m_1$ at an instant when $m_2$ has linear velocity $v_0$. For simplicity assume the masses are moving in a vertical circle and the center of mass follows a parabolic trajectory. Determine the tension in the string after the release.
Problem 10.4. Integrate Equation (10.15) directly and show that it leads to the equation of an ellipse. Note that $\alpha = -1$, $\beta$ is positive and $\gamma$ is negative.

Problem 10.5. A satellite is in an elliptical orbit around Earth. Somehow you determine that the maximum and minimum speeds of the satellite are $v_1$ and $v_2$. Find the values of $a$, $e$, the angular momentum per unit mass, and the period in terms of $v_1$, $v_2$, $G$, and $M_E$, the mass of Earth.

Problem 10.6. An interplanetary spacecraft is parked in an elliptical orbit about the Sun. The period is $\tau$. The rocket motors are fired in a short burst, and the speed of the spacecraft increases from $V$ to $V + \Delta V$. What is the change in the period?

Problem 10.7. (a) By some magical process, the Sun suddenly loses half of its mass. Show that the Earth’s orbit will be a parabola (and so the Earth will escape to infinity). (b) By some other magical process, the Sun’s mass suddenly doubles. What will the Earth’s orbital period be in this case?

Problem 10.8. Explain why the Moon raises two tides, on opposite sides of the Earth (one centered on the sub-lunar point and the other on the opposite side). It is conceptually somewhat simpler to consider the tides raised by the Sun, because the distance to the center of mass of the Sun-Earth system is essentially the same as the Sun-Earth distance whereas the center of mass of the Moon-Earth system is inside the Earth. You may ignore the daily rotation of the Earth, but not its rotation about the Sun. (Hint: Draw the vector forces acting on particles on the surface of Earth and resolve into a component parallel to the Sun-Earth line to yield the centripetal force and note the directions of the unbalanced force components.)

Problem 10.9. A certain binary star system is composed of two stars of comparable masses $m_1$ and $m_2$. (For this system we cannot assume one mass is infinitely greater than the other.) Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the positions of $m_1$ and $m_2$ relative to an inertial origin and let $\mathbf{r'}_1$ and $\mathbf{r'}_2$ be their positions relative to the center of mass. (a) Show that the primed coordinates are related to the relative coordinate $\mathbf{r}$ by

$$\mathbf{r'}_1 = -\frac{m_2}{m_1 + m_2} \mathbf{r},$$
$$\mathbf{r'}_2 = +\frac{m_1}{m_1 + m_2} \mathbf{r},$$

where the relative coordinate $\mathbf{r}$ points from $m_1$ to $m_2$. (b) Obtain the Lagrangian in terms of the relative coordinate $\mathbf{r}$ and the position of the
center of mass \( \mathbf{R} \). (c) Obtain the equations of motion in terms of the relative coordinate. (You may assume the center of mass remains at rest.) (d) Express the radial equation (for the magnitude of \( \mathbf{r} \)) in terms of the angular momentum and the reduced mass. (e) By comparing the radial equation with Equation (10.8), obtain an expression for Kepler’s third law for this situation.

**Problem 10.10.** Show that for a planet in an elliptical orbit about the Sun, the radial velocity at any time is given by

\[
r^2 \dot{r}^2 = \left[ \frac{G M}{a} \left( a[1 + e] - r \right) \left( r - a[1 - e] \right) \right].
\]

where \( M \) is the mass of the Sun. (Hint: Use Equations (10.27) and (10.25).)

**Problem 10.11.** A particle of mass \( m \) is acted upon by an attractive central force given by \( K/r^4 \). The particle is placed a distance \( a \) from the force center and given an initial velocity \( \sqrt{2K/3ma^3} \) at right angles to the radius vector. (a) Show that the particle spirals into the force center by deriving the equation for the orbit. (b) Determine the time for the particle to collide with the force center. (Hint: For this force, the potential energy is \( V = -K/3r^3 \).)

**Problem 10.12.** Consider a central force given by \( F(r) = -K/r^3 \) with \( K > 0 \). Plot the effective potential and discuss possible types of motion.

**Problem 10.13.** A particle is subjected to a central force

\[
F(r) = -\frac{K}{r^2} + \frac{K'}{r^3}.
\]

Assume \( K > 0 \) and consider both signs for \( K' \). (a) Draw the effective potential and discuss possible types of motion. (b) Solve the orbital equation and show that the bounded orbits have the form

\[
r = \frac{a(1 - e^2)}{1 + e \cos \alpha \theta}
\]

as long as \( l^2 > -mK' \).

**Problem 10.14.** The first artificial satellite of the Earth was the Russian Sputnik I. Its perigee was 227 km. above the Earth’s surface. At this point its speed was 28,710 km/hr. Determine its period of revolution. What is the maximum distance from the satellite to the surface of the Earth? Given: the Earth has a radius of 6.37X10^3 km.
**Problem 10.15.** A certain satellite has a perigee of 360 km and an apogee of 2549 km above the Earth's surface. Find its distance above the surface when it is $90^\circ$ from perigee, as measured from the center of Earth.

**Problem 10.16.** For the two body problem there is a conserved quantity called the Laplace vector. (It is also called the Runge-Lenz vector.) The Laplace vector is a vector pointing towards periapsis. Its magnitude is proportional to the eccentricity. It can be expressed as:

$$ A = p \times l - GMm^2 \hat{r} $$

where $M$ is the mass of the primary, $p$ is the linear momentum and $l$ is the angular momentum.

a) Show that $A$ lies in the orbit plane.

b) Show that $A$ is a constant of the motion.

c) Show that the magnitude of $A$ is $GMm^2 e$. (Hint: Evaluate $r \cdot A$.)

**Problem 10.17.** Imagine a circular orbit passing through the origin. Then $r = A \cos \theta$. Show that a central force that gives rise to such an orbit has a $1/r^5$ dependence.

**Problem 10.18.** A particle moves under the action of a central force in a spiral described by $r = Ae^{b\theta}$. Show that the force acting on it is inversely proportional to the cube of $r$.

**Problem 10.19.** A satellite is in a circular orbit, a distance $r$ from the center of Earth. It has a known velocity $v$. Show that this satellite has an escape velocity (from $r$ to $\infty$) of $\sqrt{2}v$.

**Problem 10.20.** An asteroid is in a circular orbit at 5 AU from the Sun. Express its speed ($v$) in terms of the mass of the Sun and other constant parameters. For some reason, space scientists of the future decide that the asteroid should be ejected from the Solar System. What is the minimum velocity required to do so? (Ignore the effect of the planets.)

**Problem 10.21.** Two stars of equal mass orbit around their common center of mass with period $\tau$. Show that if they were suddenly stopped, dead still, and allowed to fall toward each other, they would collide in a time $\tau/\sqrt{32}$.

**Problem 10.22.** Consider a binary star system. (a) Show that Kepler’s third law can be written as

$$ a^3 = \tau^2 (M_1 + M_2) $$
where \( a \) is in AU (astronomical units) \( \tau \) is in Earth years and \( M_1 + M_2 \) is the sum of the masses of the stars in solar masses. (Note: All the equations we developed in this chapter assumed \( M \gg m \). Hint: Rederive Equation (10.1) and show that if the two masses are comparable, that \( GM \) should be replaced by \( G(M_1 + M_2) \) everywhere.)

(b) Star A and star B are members of a binary star system. An astronomer determines that the period of this system is 32 years and that the stars are separated by 16 AU. Furthermore, star A is found to be 12 AU from the center of mass of the system. Determine the masses of these stars. (Note: This is one of the ways in which astronomers evaluate the masses of stars.)

**Problem 10.23.** The Rutherford problem consists in the motion of an alpha particle interacting with a gold nucleus. The force is a repulsive force of magnitude \( F = K/r^2 \).

(a) Show that the eccentricity can be expressed as

\[
e^2 = 1 + \frac{2El^2}{mK^2}.
\]

(b) Show that the distance of closest approach is

\[
r_1 = a(e + 1).
\]

(Hint: Recall that for a repulsive force the orbit is the “minus branch” of the hyperbola.)

**Problem 10.24.** A body is moving in an elliptical orbit. Its maximum velocity is \( v_{\text{max}} \) and its minimum velocity is \( v_{\text{min}} \). Express the eccentricity of the orbit in terms of \( v_{\text{max}} \) and \( v_{\text{min}} \).

**Problem 10.25.** Assume the space shuttle is in a circular orbit about Earth with its nose pointing forward. The rocket motors are fired for a few moments. You would expect the shuttle to speed up. Show that, instead, it actually rises to a higher orbit and slows down. (This behavior is called the “satellite paradox.”)

**Problem 10.26.** Assume the force between two particles is a central attractive force proportional to the separation between the particles. That is, \( \mathbf{F} = -kr\hat{r} \). You can assume one of the particles is much more massive than the other one. Show that the motion of the lighter mass is an ellipse with the more massive particle at the center of the ellipse (rather than at a focal point).

**Problem 10.27.** The Moon raises tides on Earth, causing the Earth’s rotation rate to decrease and causing the radius of the Moon’s orbit to increase. (See the discussion accompanying Figure 7.3.) According to Kepler’s third law, the angular velocity of the Moon will
10.13. PROBLEMS

Problem 10.28. One of the strongest arguments made for the general theory of relativity is that it predicts the correct value for the precession of the orbit of the planet Mercury. Before Einstein formulated that theory, it was suggested that the precession could be explained if the solar system were filled with dust having a low density $\rho$. The spherical distribution of dust would exert an additional central force on a planet given by $F' = -mKr$ where $m$ is the mass of the planet and $K$ is a constant equal to $(4\pi/3)pG$. Show that for $F' \ll F$ where $F$ is the gravitational force of the Sun, a planet will move in an elliptical orbit whose major axis precesses at an angular velocity

$$\omega_p = \frac{2\pi \rho r_0^{3/2}}{\sqrt{G/M}},$$

where $M$ is the mass of the Sun, and $r_0$ is the average radius of the orbit.

Problem 10.29. Consider an imaginary central force $f(r)\hat{r}$ which has the property that all circular orbits around the force center have the same areal velocity. Determine $f(r)$.

Problem 10.30. Consider a particle that is acted upon by the central force:

$$F = \frac{A}{r^2} + \frac{B}{r^4},$$

where $A$ and $B$ are constants. The orbit is a circle of radius $a$. Determine the condition for the orbit to be stable.

Problem 10.31. A particle is moving in a circular path under the action of an attractive central force given by

$$F = \frac{1}{r^2}e^{-r/a}.$$  

Show that if the radius of the circle is greater than $a$ the motion is stable, and if the radius of the circle is less than $a$, the motion is unstable.

Problem 10.32. An ideal massless string passes through a small hole in a perfectly smooth table. Two equal masses are attached to the
ends of the string so that one mass is on the table a distance $a$ from the hole, and the other mass is hanging freely. See the sketch (Figure 10.15). The mass on the table is set in motion with a velocity $\sqrt{ga}$ perpendicular to the direction of the string. Show that the mass on the table moves in a circular path. Now the hanging mass is perturbed slightly so that it oscillates up and down. Show that the period of this oscillation is $2\pi\sqrt{2a/3g}$.

Figure 10.15. Mass on frictionless table (see Problem 10.32)

Problem 10.33. Consider a particle moving in an attractive inverse cube force law. That is, $\mathbf{F} = -(c/r^3)\hat{r}$ where $c$ is a positive constant. (a) Show that if the particle is slightly perturbed, the zeroth, first, and second order terms in the equation for $\ddot{\mathbf{r}}$ are zero. (b) Assuming that you can generalize the results of part (a) to $\ddot{\mathbf{r}} = 0$, describe what you expect for the subsequent motion of the perturbed particle. (c) You can get a good idea of the motion by plotting the effective potential for this problem. Show that the particle will either spiral in to the origin or spiral out to infinity. (The curve $r = r(\theta)$ is called a Cotes spiral.)

Problem 10.34. Given, $f(r) \propto 1/r^n$. Show that closed orbits exist for $n = -2$ and $n = 1$, but not for $n = 2$ or $n = 3$. (Hint: For a closed orbit, the apsidal angle must be a rational fraction of $2\pi$. The apsidal angle is the angle through which the radius vector rotates between apsides.)

COMPUTATIONAL PROJECTS

Computational Techniques: The Runge-Kutta Method

In Section 3.7 I described the Euler Cromer algorithm for solving ordinary differential equations (ODE’s). Here I will describe the Runge-Kutta method which is more accurate and perhaps more elegant, but
less transparent. It is based on the truncated Taylor Series expansion for a function \( g(t) \):

\[
g(t + \tau) = g(t) + \tau \frac{dg}{dt} \bigg|_{\xi}
\]

The second term is usually evaluated at \( t \), but to make the solution more accurate, the Runge-Kutta technique evaluates it half way through the time step. That is \( \xi = t + \tau/2 \).

An economical way of expressing the ODE is to define two vectors, \( \mathbf{x} \) and \( \mathbf{f} \) in the following way. Assume a two dimensional system so the position is given by \( x \) and \( y \) and the velocity by \( v_x \) and \( v_y \). Then

\[
\mathbf{x}(t) = [x(t) \ y(t) \ v_x(t) \ v_y(t)]
\]

\[
\mathbf{f}(\mathbf{x},t) = [v_x(t) \ v_y(t) \ a_x(t) \ a_y(t)]
\]

and the ODE can be written

\[
\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), t).
\]

The second-order Runge-Kutta algorithm is obtained by first defining a new vector \( \mathbf{x}^* \) by:

\[
\mathbf{x}^* = \mathbf{x}^*(t + \tau/2) = \mathbf{x}(t) + \frac{1}{2} \tau \mathbf{f}(\mathbf{x}(t), t).
\]

Then

\[
\mathbf{x}(t + \tau) = \tau \mathbf{f}(\mathbf{x}^*, t + \tau/2).
\]

It is not too difficult to appreciate that this is equivalent to the truncated Taylor Series. However, the most common ODE solver is not the second-order Runge-Kutta, but the fourth-order Runge-Kutta, which is also based on the Taylor Series but in a much less transparent way. In vector form, the value of \( \mathbf{x} \) at time \( t + \tau \) is given by

\[
\mathbf{x}(t + \tau) = \mathbf{x}(t) + \frac{1}{6} \tau (\mathbf{F}_1 + 2\mathbf{F}_2 + 2\mathbf{F}_3 + \mathbf{F}_4),
\]

where

\[
\mathbf{F}_1 = \mathbf{f}(\mathbf{x}, t),
\]

\[
\mathbf{F}_2 = \mathbf{f}(\mathbf{x} + \frac{1}{2} \tau \mathbf{F}_1, t + \frac{1}{2} \tau),
\]

\[
\mathbf{F}_3 = \mathbf{f}(\mathbf{x} + \frac{1}{2} \tau \mathbf{F}_2, t + \frac{1}{2} \tau),
\]

\[
\mathbf{F}_4 = \mathbf{f}(\mathbf{x} + \frac{1}{2} \tau \mathbf{F}_3, t + \tau).
\]
Computational Project 10.1. A comet is jostled loose from the Oort Cloud and heads toward the Sun. Suppose that when it passes the orbit of Pluto it has velocity components \( v_x = -0.01 \), \( v_y = 0.05 \) AU/year. Plot the trajectory of the comet using three algorithms: Euler-Cromer, Second Order Runge Kutta and Fourth Order Runge Kutta. Show that the angular momentum is constant. (The only bodies involved are the comet and the Sun. Hint: Use \( GM = 4\pi^2 \).)

Computational Project 10.2. This problem is the same as the previous problem, but now include the effect of Jupiter (which you can assume has a circular orbit). The trajectory of the comet lies in the plane of the orbit of Jupiter. Use RK4.

Computational Project 10.3. Write a program to trace out the positions of the planets as a function of time, using the information in the following table. You may assume the planets have zero inclination (so they all move in the same plane). Start the planets so that they are all lined up in a straight line from the Sun. (This is called a “conjunction”.) (a) Determine the number of years between the conjunctions of Jupiter and Saturn. (b) What fraction of the time is Pluto nearer the Sun than Neptune? (P.S. I know Pluto is no longer a planet.)

<table>
<thead>
<tr>
<th>Planet</th>
<th>a (AU)</th>
<th>e</th>
<th>Mass ((10^{24}\text{kg}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>0.387</td>
<td>0.206</td>
<td>0.33</td>
</tr>
<tr>
<td>Venus</td>
<td>0.723</td>
<td>0.007</td>
<td>4.87</td>
</tr>
<tr>
<td>Earth</td>
<td>1.000</td>
<td>0.017</td>
<td>5.97</td>
</tr>
<tr>
<td>Mars</td>
<td>1.524</td>
<td>0.093</td>
<td>0.64</td>
</tr>
<tr>
<td>Jupiter</td>
<td>5.203</td>
<td>0.048</td>
<td>1898.6</td>
</tr>
<tr>
<td>Saturn</td>
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<td>0.054</td>
<td>568.46</td>
</tr>
<tr>
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<td>0.047</td>
<td>86.83</td>
</tr>
<tr>
<td>Neptune</td>
<td>30.069</td>
<td>0.009</td>
<td>102.43</td>
</tr>
<tr>
<td>Pluto</td>
<td>39.482</td>
<td>0.249</td>
<td>0.013</td>
</tr>
</tbody>
</table>

Computational Project 10.4. Compute and plot the motion of a particle under the action of the central force

\[
F = -\frac{K}{r^3}(1 - \frac{\alpha}{r})r
\]

where \( K \) and \( \alpha \) are constants. Show that this orbit precesses. Show how your choice of \( K \) and \( \alpha \) affect the motion.

Computational Project 10.5. We consider the effect of atmospheric drag on an artificial Earth satellite. The drag force acts tangentially to the orbit. This force can be expressed as

\[
F = -\frac{1}{2m}C_D A \rho v^2,
\]
where $m$ = mass of the satellite, $C_D$ = the drag coefficient, $A$ = the cross sectional area of the satellite, $\rho$ = the air density, and $v$ = the velocity of the satellite. The density of air at altitude $\eta$ is given approximately by
\[
\rho = \rho_0 \exp \left[ -(\eta - \eta_0)/H \right]
\]
where $\rho_0 = 6.5 \times 10^{-12}$ kg/m$^3$, $\eta_0 = 384$ km and $H$ is a “scale height” for which we can use 40 km at altitudes above about 100 km.

It is not difficult to show that the effect of drag on the semi-major axis and the eccentricity of the orbit are given by the expressions
\[
\Delta a = -\frac{A}{m} C_D a^2 \int_0^{2\pi} \rho(\eta) \left( \frac{1 + e \cos E}{1 - e \cos E} \right)^{3/2} \cos EdE
\]
\[
\Delta e = -\frac{A}{m} C_D a (1 - e^2) \int_0^{2\pi} \rho(\eta) \left( \frac{1 + e \cos E}{1 - e \cos E} \right)^{1/2} \cos EdE
\]
where $E$ is a parameter called the eccentric anomaly (see Computational Project 10.6). Evaluate $\Delta a$ and $\Delta e$ by numerically integrating the given expressions. You may assume $C_D = 1.05$, $A = 6$ m$^2$, and $m = 1000$ kg. Note: The distance from the center of Earth to the satellite can be expressed in terms of the eccentric anomaly by
\[
r = a(1 - e \cos E).
\]

**Computational Project 10.6.** In Exercise (10.12) you evaluated the average potential energy of a planet in orbit to be $-GMm/a$ by simply averaging the maximum and minimum values of $r$. You are now asked to show that the time average of potential energy is given by that expression. You should determine the potential energy at equal time steps as the planet goes around in orbit. For simplicity let the planet be at 1 AU, have a period of one year and an eccentricity $e = 0.2$. The problem is complicated by the fact that the planet does not orbit at a constant angular speed. However, you can get the position of the planet at equal intervals of time by determining the eccentric anomaly $E$ which is related to the average angular velocity $n$ by Kepler’s equation\(^{13}\)
\[
E - e \sin E = nt.
\]
This is a transcendental equation so you will have to solve for $E$ numerically. Write a program that obtains $E$ iteratively, using the fact

\(^{13}\)The eccentric anomaly is a parameter that is well known by celestial mechanicians and astronomers. Kepler’s equation is a famous equation whose analytical solution has not been found. You can read about it in Chapter 2 of C. D. Murray and S. F. Dermott, *Solar System Dynamics*, Cambridge University Press, Cambridge, 1999.
that $e$ is a small quantity. That is, write Kepler’s equation as

$$E = nt + e \sin E.$$  

Since $e$ is small, the first approximation to $E$ is $E = nt$. Use this for $E$ on the right hand side and get the second approximation. Repeat over and over. Thus:

$$E_0 = nt$$
$$E_1 = nt + e \sin E_0$$
$$E_2 = nt + e \sin E_1$$

etc.

To solve for the average potential energy, let the planet orbit the Sun 10 times, determine the time average potential energy and compare it with $-GMm/a$. 